THE

CONSTRUCTION AND ANALYSIS

OF

GEOMETRICAL PROPOSITIONS,

DETERMINING THE POSITIONS ASSUMED BY HOMOGENEAL BODIES WHICH FLOAT FREELY, AND AT REST, ON A FLUID'S SURFACE;

ALSO DETERMINING THE

STABILITY OF SHIPS, AND OF OTHER FLOATING BODIES.

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FROM THE

PHILOSOPHICAL TRANSACTIONS.

THE CONSTRUCTION, &c.

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To investigate the positions assumed by homogeneal bodies which float freely, and at rest, on a fluid's surface, it is necessary, in the first place, to form a just conception of the several principles on which those positions depend.

The proportion of the immersed part to the whole magnitude of a floating body * will always be obtained, from having given the specific gravity of the solid in respect to that of the fluid; since it is a known law of hydrostatics, that the immersed part of the solid is to the whole magnitude, in the proportion of those specific gravities. But a solid may be immersed in a fluid numberless different ways, so that the part immersed shall be to the whole magnitude in the given proportion of the specific gravities, and yet the solid shall not rest permanently in any of these positions. The reasons are obvious. The floating body is impelled downward by its weight, acting in the direction of a vertical line, which passes through the centre of gravity; the pressure of the fluid, by

* In these pages the floating bodies are always understood to be homogeneal, unless the contrary be mentioned.

which the solid is supported, acts upward, in the direction of a vertical line (usually called the line of support), which passes through the centre of gravity of the part immersed: unless, therefore, these two lines are coincident, so that the two centres of gravity shall be in the same vertical line, it is evident that the solid thus impelled, must revolve on an axis until it finds a position in which the equilibrium of floating will be permanent.

From these observations it appears, that to ascertain the positions in which a solid body floats permanently on the surface of a fluid, it is requisite that the specific gravity of the floating body should be known, in order to fix the proportion of the part immersed to the whole: secondly, it is necessary to determine, by geometrical or analytical methods, in what positions the solid can be placed on the surface of the fluid, so that the centre of gravity of the floating body, and that of the part immersed, may be situated in the same vertical line, while a given proportion of the whole volume is immersed under the fluid's surface.

These particulars having been determined, evidently reduce the statement of the problem into a narrow compass; but they are not alone sufficient to limit it: for although it has been shewn that a body cannot float permanently on a fluid unless the two centres of gravity, that have been mentioned, are situated in the same vertical line, it does not follow that, whenever those centres of gravity are so situated, the solid will float permanently in that position:* consistently

* Admitting any proposition to be true, the converse of the proposition may be either true generally, or with exceptions. To distinguish the cases in which it is true from those in which it fails, requires a separate demonstration or investigation.

with this observation, positions may be assigned, in which a solid is immersed in a fluid to the true depth according to its specific gravity, and the centre of gravity of the solid and that of the part immersed are in the same vertical line, yet the solid does not rest in any of these positions, but assumes some other in which it will continue permanently to float. To make this evident, a very obvious instance may be referred to. Suppose a cylinder, the specific gravity of which is to that of a fluid on which it floats as 3 to 4; and let the axis of the cylinder be to the diameter of the base as 2 to 1: if this cylinder is placed on the fluid with its axis vertical, it will sink to a depth equal to a diameter and a half of the base; and as long as the axis is sustained in a vertical position by external force, the centre of gravity of the solid, and the centre of the immersed part, will be situated in the same vertical line: but the solid will not float permanently in that position; for as soon as external support is removed, it falls from its upright position, and remains floating with the axis horizontal. If the axis of the cylinder is made only $\frac{1}{2}$ instead of twice the diameter of the base, the solid being placed with its axis vertical, will sink to the depth of $\frac{3}{8}$ of a diameter, and will float permanently in that position. Even if the axis should be placed not exactly coincident with the vertical, but in a direction somewhat inclined to that line, the solid will change its position until it settles permanently with the axis perpendicular to the horizon.

The cylinder here instanced is caused either to float permanently with its axis vertical, or to overset, according to the different proportions between the length of the axis and the diameter of the base: although an exact estimate of the effects

produced by altering these proportions, cannot be obtained except by mathematical investigation (a subject to be considered in some of the following pages), yet a general idea of the causes by which so remarkable a difference is occasioned in the floating position of the two cylinders, will appear obvious by attending to the changes which take place in the position of the line of support, while the solid is inclined from the upright through a small angle. For whenever the line of support, in the direction of which the force of the fluid's pressure acts, does not pass through the centre of gravity of the floating body, that force must generate a motion of rotation round an horizontal axis which passes through the centre of gravity of the solid; and must cause an elevation of those parts of the solid which are on the same side of the axis of motion with the line of support, and consequently must depress those parts which are situated on the contrary side of that axis. Admitting, therefore, that the solid is adjusted with its centre of gravity and the centre of the immersed part precisely to the same vertical line, and that a small inclination takes place round the axis of motion; it will depend on the position of the line of support, whether that inclination shall be counteracted, so as to restore the solid to its upright position, or shall be augmented; in which latter case the solid oversets. If the nature of the figure should be such as causes the line of support to be moved toward those parts which are immersed by the inclination, that inclination will be counteracted, because the pressure of the fluid generates angular motion in a direction contrary to that in which the solid is inclined; but if the figure is such as causes the line of support to be moved toward those parts of the solid which

are elevated by the inclination, the force of the fluid's pressure must continually augment the inclination; or, in other words, will cause the solid to overset, or change its position, until it settles in some other, in which the equilibrium is permanent.

We observe, therefore, that a solid floats permanently in a given position, only because the smallest inclination from that position creates a force by which the inclination is immediately counteracted, and the solid becomes restored to its upright position; and consequently, since the inclination is counteracted while of evanescent magnitude, no sensible deviation from the upright can take place: in cases of instability, the solid oversets, although placed on a fluid with the centre of gravity of the solid and that of the part immersed in the same vertical line, because the smallest deviation or inclination from that position creates a force by which the inclination is augmented. And since various causes concur in preventing the two centres from remaining adjusted to the vertical with a precision absolutely mathematical, it follows that the least or evanescent inclination here mentioned must necessarily subsist, and being continually augmented by the fluid's pressure, must become a sensible rotation, by which the solid oversets from its upright position.

In either case, that is, whether the solid floats permanently, or oversets, if it is placed on the surface of a fluid, so that the centre of gravity of the solid and the centre of gravity of the part immersed shall be in the same vertical line, the solid is said to be in a position of equilibrium: and from the preceding observations it appears, that there are three species of equilibrium in which a solid may be situated when the two centres of gravity just mentioned are in the same vertical line.

1st.* The equilibrium of stability, in which the solid floats permanently in a given position.

2dly. The equilibrium of instability, in which case the solid, although its centre of gravity and that of the part immersed are in the same vertical line, spontaneously oversets, unless sustained by external force. This kind of equilibrium is similar to that which subsists when a needle, or other sharp-pointed body, is placed vertically on a smooth horizontal surface.

3dly. The third species, being a limit between the two former, is called the equilibrium of indifference, or the insensible equilibrium, in which the solid rests on the fluid indifferent to motion, without tendency to right itself when inclined, or to incline itself further.

These different kinds of equilibrium may perhaps be more clearly perceived, by referring to the instance in which a cylinder was supposed to be placed on the surface of a fluid with the axis vertical. If the axis is assumed double the diameter of the base, the solid oversets, the equilibrium of position being that of instability: but if the length of the axis is only half the diameter of the base, the solid floats permanently with the axis vertical. It seems evident, therefore, that there must be some intermediate proportion between the cylinder's axis and the diameter of the base, greater than 1 to 2, and less than 2 to 1, which will correspond to the case intermediate, where stability ceases, and instability begins: this is the precise proportion when the equilibrium is of the species called the equilibrium of indifference, or the insensible equilibrium.

When a solid body floats permanently on the surface of a

* Euler. *Théorie complette de la Construction et Manœuvre des Vaisseaux*, chap. iv.

fluid, and external force is applied to incline it from its position, the resistance opposed to this inclination is termed the stability of floating. It is obvious to every one's experience, that some floating bodies are more easily inclined from their quiescent position than others; that, after having been inclined, some will return to their original situation with more force and celerity than others; a difference particularly observable in ships at sea, in some of which a given impulse of the wind will cause a much greater inclination from the perpendicular than in others. As this property of opposing resistance to heeling or pitching, when regulated to its due quantity and proportion, has been deemed of material consequence in the construction of vessels, several eminent mathematicians have been induced to investigate rules, by which the stability of ships may be inferred, independently of any reference to trial, from knowing their weights and dimensions only. It must, however, be acknowledged, that the theorems which have been given on this subject, in the works of Mons. BOUGUER,* EULER,† FRED. CHAPMAN,‡ and other writers, for determining the stability of ships, are founded on a supposition that the inclinations from their quiescent positions are evanescent, or, in a practical sense, very small. But as ships at sea are known to heel through angles of 10°, 20°, or even 30°, a doubt may arise how far the rules demonstrated on the express condition, that the angles of inclination are of evanescent magnitude, should be admitted as practically applicable in cases where the inclinations are so great.

* BOUGUER. Liv. i. sec. iii. chap. iv.

† EULER. *Théorie complette de la Construction et Manœuvre des Vaisseaux*, chap. iv. and chap. v.

‡ *Traité de la Construction des Vaisseaux par* FRED. CHAPMAN, chap. ii. p. 17.

To put this matter in a clear point of view, let a case be assumed. Suppose two vessels to be of the same weight and dimensions in every respect, except that the sides of one of these vessels shall project more than those of the other, the projections commencing from the line coincident with the water's surface. According to the theorems of Bouguer and other writers, the stability will be the same in both ships, which is in fact true, on the supposition that their inclinations from the perpendicular are extremely small angles: but when the ships heel to 15° or 20°, the stabilities of the two vessels must evidently be very different. Even supposing the stability of a ship A to be greater than that of a ship B, when the angles of heeling are very small, it may happen in cases easily supposable that when both ships are heeled to a considerable angle of inclination, the stability of the ship B shall exceed that of the ship A. Admitting, therefore, that the theory of statics can be applied with any effect to the practice of naval architecture, it seems to be necessary that the rules investigated for determining the stability of vessels should be extended to those cases in which the angles of inclination are of any magnitude likely to occur in the practice of navigation.

When a solid is placed on the surface of a lighter fluid, at the proper depth corresponding to the relative gravities, it cannot change its position by the combined actions of its weight and the fluid's pressure, except by revolving on some horizontal axis which passes through the centre of gravity. Various axes may be drawn through the centre of gravity of a floating body in a direction parallel to the horizon: but since the motion of the solid respecting one axis only, can be the subject of the same investigation (except in extreme cases

not to be considered in this place), the figure of the floating body, and the particular object of inquiry, must determine to which of these axes the motion of the solid is to be referred, when it changes its position: thus, suppose a square beam of timber, the specific gravity of which is to that of water as 1 to 2, should be placed on the surface of that fluid with one of the flat surfaces parallel to the horizon (the length being assumed considerably greater than the breadth), no motion of rotation can take place round the transverse axis, by which the extremities of the beam would be elevated or depressed: but the solid will spontaneously revolve in this instance round the longer axis, changing its position until it settles with an angle upward.

In like manner, if the same solid should be placed horizontally on the surface of the water with an angle upward, it will not spontaneously change its position; but if one extremity of the beam should be forcibly elevated, and the other depressed, so as to incline the longer axis to the horizon, as soon as all external force is removed, the beam will revolve on a transverse horizontal axis, passing through the centre of gravity, and perpendicular to the longer axis, until it settles in such a position as to leave the longer axis horizontal. These are instances in which the figure of the body, and the particular nature of the case, determine the axis round which the solid revolves, while it changes its situation on a fluid's surface; this axis is called, for the sake of distinction, the axis of motion.

The axis of motion, round which the solid revolves, having been determined, and the specific gravity being known, it appears from the preceding observations, that the positions of permanent floating will be obtained, first by finding the

several positions of equilibrium through which the solid may be conceived to pass, while it revolves round the axis of motion; and secondly, by determining in which of those positions the equilibrium is permanent, and in which of them it is momentary and unstable.

In proceeding to investigate the principles which are the objects of the present inquiry, it will be convenient in the first instance to consider the floating body to be some homogeneal solid of regular figure, and uniform shape and dimensions, in respect of the axis of motion throughout. If such a solid is supposed to be cut through by vertical planes in a direction perpendicular to the axis of motion, the sections of these planes with the solid will be areas precisely equal and similar. Let EDHF (Tab. III. fig. 1.) represent the vertical section of such a solid, which passes through the centre of gravity G in a direction perpendicular to the axis of motion. The solid floats on the surface of a fluid IABL; consequently ADHB represents the part immersed under the fluid's surface; O is the centre of gravity of the part immersed, and the line GOC is assumed perpendicular to the horizontal line AB. We are in the next place to suppose that this solid is inclined round the axis of motion from its former position through an angle KGS (fig. 2.);* so that the line KC which was before ver-

* When this inclination takes place, the centre of gravity G, through which the axis of motion passes, is not necessarily fixed, but must evidently in most cases change its place, since the total volume immersed before the inclination is always equal to that which is immersed after the inclination; and from this cause such change of place ensues: but the motion of the axis, and of the point G, is wholly independent of the reasoning in this and the subsequent constructions and investigations; the object of which is to ascertain the angular motion round the said axis, and the several consequences thereof, and is no ways connected with the motion of the axis itself. This note

tical, may be now transferred to the position SL, which is inclined to the vertical line KC at the angle KGS: moreover the line AB, which was before horizontal, is transferred so as to coincide with the line IN, being inclined to its former position in the angle NXP, which is equal to KGS: and consequently the whole space ADHB, becomes transferred so as to coincide with the space IRMN, and the volume immersed under the fluid's surface is WRMNP. If in the line SL, GE is taken equal to GO; it is evident that in consequence of the inclination, the point O, which is the centre of gravity of the space ADHB, will be transferred to the point E, which is the centre of gravity of the equal space IRMN; and the pressure of the fluid would act on the solid in the direction of a vertical line passing through the point E, if the space IRMN was the volume immersed under the fluid's surface; but in consequence of the inclination of the solid through the angle KGS, the volume NXP, which was before above the fluid's surface, will now become immersed under it; and the volume IWX, which was before under the surface, will become elevated above it. It is evident, that on both these accounts, that is, both by the addition of the volume NXP, and the abstraction of the volume IWX, the centre of gravity E of the space IRMN will be transferred towards those parts of the solid which have become more immersed under the fluid in consequence of the inclination.

Suppose the centre of gravity of the volume immersed, WRMP, to be situated at the point Q: through Q draw

is here inserted in preference to adapting the construction so as to express the alteration in the position of the axis, which would only have the effect of embarrassing the construction with useless lines.

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QS parallel to GO; through E draw EY perpendicular to SQ; and through G draw *z*GZ perpendicular to SQ. Then, since the point Q is the centre of gravity of the part immersed, the pressure of the fluid will act in the direction of the vertical line QS, with a force equal to the body's weight, and by the principles of mechanics will have precisely the same effect to turn the solid round its axis as if the same force was applied immediately at the point Z, acting in the same direction QS. Since, therefore, the effect of the fluid's pressure acting in the direction of a vertical line which passes through the centre of gravity Q, no way depends on the absolute position of that point, but on the perpendicular distance GZ, between the two vertical lines GO and SQ only, in proceeding to ascertain, by geometrical construction, the several positions which bodies assume on a fluid's surface, and their stability of floating, the determination of the absolute position of the point Q, or centre of gravity of the immersed part, will not be necessary; the perpendicular distance GZ between the two vertical lines which pass through the centres of gravity of the solid, and of the part immersed, being sufficient for obtaining all the results that are required.

The part immersed, before the inclination of the solid took place, is ADHB; when the solid has been inclined through the angle KGS, the part immersed is WRMP, which is the volume IRMN diminished by the space IWX, and augmented by the space NXP. But since the volume immersed under the fluid's surface must always be of the same magnitude while the solid's weight continues unaltered, it follows, that whatever additional space is immersed under the surface in consequence of the inclination, an equal space must be ele-

vated above it; consequently, whatever may be the position of the point of intersection X, the volume I X W must be equal to the volume P X N. Suppose *a* to be the centre of gravity of the space I X W, and let *d* be the centre of gravity of the space N X P; then, the part immersed W R M P, is equal to the space IRMN, diminished by the space IWX, considered as concentered in the point *a*, and increased by the equal space N X P, concentered in the point *d*; consequently the centre of gravity Q of the space W R M P will be at such a distance from E, the centre of gravity of the space IRMN, as corresponds to the alteration occasioned by removing the volume I W X, concentered in the point *a*, to the point *d*. These are the data from which the perpendicular distance G Z, of the two vertical lines K O, S Q, passing through the centres of gravity G and O, is to be obtained in the manner following: through the centres of gravity *a* and *b*, draw the lines *ab*, *dc*, perpendicular to the horizontal line AB; through E draw the indefinite line EY parallel to AB, and in the line EY, take a part ET, so that ET shall be to the line *bc* as the volume IWX, or its equal NXP, is to the whole volume immersed, WRMP or ADHB: through the point T thus found, draw the line F T S parallel to the vertical line G O; the centre of gravity Q, of the immersed part, will be somewhere in the line FS; and because ER is to E G, as the sine of the given angle of inclination is to radius, the line GO = EG being supposed given, the line ER will therefore be known, and being subtracted from the line ET before found, will leave RT or GZ the perpendicular distance between the two vertical lines, which it was required to determine by geometrical construction, and which has been accordingly determined.

The demonstration of this construction is founded on an obvious and elementary principle of mechanics.—It is this.—The common centre of gravity of any system of bodies (considered as heavy points or centres of gravity), being given in position, if one of these bodies should be moved from its place, the corresponding motion of the common centre of gravity, estimated in any given direction, will be to the motion of the aforesaid body, estimated in the same direction, as the weight of the body moved is to the weight of the whole system. To apply this proposition. The volume IRMN (fig. 2.) may be assumed as a system of bodies, of which the common centre of gravity is E. One of the bodies composing this system, namely, the volume I W X, concentered in the point *a*, is transferred in consequence of the inclination of the solid through the angle S G K from the point *a* to the point *d*, in which the equal volume N X P is concentered: this will have the effect of moving the common centre of gravity of the system E. But it is required to find how much the position of this centre E has been changed in the direction EY parallel to A B, which is the given direction stated in the proposition. The motion of the centre of gravity *a*, from *a* to *d*, estimated in the given horizontal direction, is *bc:* then, according to the mechanical proposition, as the volume W R M P or A D H B is to the volume I W X or N X P, so is the line *bc* to ET, the corresponding motion of the centre of gravity E estimated in the given horizontal direction; consequently if a line FTS is drawn through the point T parallel to the vertical line GO, the centre of gravity of the immersed part Q must be situated somewhere in the line FTS: subtracting from ET the line E R (which is the sine of the given

angle of inclination EGO when EO is the radius), there will remain the line RT or GZ, which is therefore the distance between the vertical lines GO, SZT, passing through the centres of gravity G and Q, as determined by the construction.

Let the whole volume of the immersed part of the solid be denoted by the letter V; suppose the space NXP, or volume immersed in consequence of the inclination, to be A; make $GO = d$; and the sine of the angle of inclination KGS (to radius 1) $= s$; also make $bc = b$. Then since by the proposition; as $b : ET :: V : A$, it appears that $ET = \frac{b \times A}{V}$;

And since as $ER : EG = GO :: \text{so is } s : 1$, we obtain $ER = ds$;

Wherefore $RT = ET - ER = \frac{bA}{V} - ds = GZ$.

This result is founded on a supposition that the figure of the floating solid is uniform in respect of the axis of motion; if the solid should be of an irregular form, the construction and demonstration will be precisely the same as in the preceding case, the following particulars being attended to; the volume, or space immersed in consequence of the inclination, will no longer be represented by the area NXP, but must be obtained by a calculation founded on the shape and dimensions of the said volume; moreover the centres of gravity of the volumes PXN, IXW, will not now correspond with the centres of gravity of the areas PXN, IXW, and must therefore be obtained from the known rules, or from methods of approximation by which the position of the centre of gravity is determined in solid bodies.

The angle of inclination KGS is given by the supposition, and the solid contents of the equal volumes denoted by IXW, NXP, with the distance *bc* of the centres of gravity *a* and *d*,

estimated in the direction of the horizontal line AB, having been determined, let the volume NXP be put $= A$; and $bc = b$; the other quantities signifying as before; the perpendicular distance $GZ = \frac{bA}{V} - ds$, will be known. It is to be observed, that this proposition in general is equally applicable to heterogeneal bodies as to those which are homogeneal.

By this proposition the stability of vessels, and other bodies floating on a fluid's surface, at any angle of inclination, from a given position of equilibrium, is obtained. For the measure of the stability is precisely a force equal to the fluid's pressure; that is, equal to the vessel's weight,* applied perpendicularly at the distance GZ from the axis of motion, to incline the solid round that axis.

From the same proposition, the different positions assumed by bodies which float freely on a fluid's surface, may be ascertained; in some cases most easily by geometrical construction; in others, by analytical investigation. It has been already observed, that to ascertain the various positions in which a body will float permanently on the surface of a fluid, it is necessary first, to have given the ratio of the specific gravities, in order to fix the proportion of the part immersed to the whole; and secondly, the several positions are to be ascertained in which the solid may rest on the surface of a fluid, so that the centres of gravity of the solid and of the part immersed may be in the same vertical line. The general expression for the line RT (fig. 2.) or GZ, is $GZ = \frac{bA}{V} - ds$; by putting this quantity $\frac{bA}{V} - ds = 0$, an equation arises, from which one or more values of s will be obtained $=$ the sine of the angle through which the solid has been inclined from a position of

* The weight of a vessel implies the weight of the ship and lading.

equilibrium, when the line $GZ = 0$; that is, when the two centres of gravity, G and Q, are again situated in the same vertical line; or in other words, when the solid is again in a position of equilibrium. By this method of proceeding, the several positions of equilibrium may be determined; it only, therefore, remains to discover in which of these positions the equilibrium is permanent, and in which of them it is momentary and unstable. This circumstance will depend on the species of equilibrium in which the solid is originally placed previously to the inclination, which, for the sake of more clearly stating the principles of stability, may be supposed known, although the rules for ascertaining this point have not yet been considered, but will appear in the pages which next follow. Assuming then the species of equilibrium, in which a solid is originally placed on the surface of a fluid, to be known, let that equilibrium be supposed permanent, or the equilibrium of stability; and let the solid be conceived to be inclined round the axis of motion, through a given angle A, till it becomes situated again in a position of equilibrium; in which case the centres of gravity of the solid, and of the part immersed, will again be in the same vertical line. Since during this inclination, the fluid's pressure acts with a force proportional to the line RT or GZ, (fig. 2.) to diminish the angular distance from the original position of equilibrium, it follows that the same force must act on the solid, so as to augment the inclination, or angular distance from the second position of equilibrium, in which the solid is situated after it has revolved through the entire angle A, or any part thereof, from its original situation; from which observations it is evident, that the second position of equilibrium must be that of instability:* and by the same

* It appears from the observations in page 4, that whenever a solid floats in a posi-

mode of argument it is shewn, that if the original position of equilibrium be that of instability, when the solid by revolving on its axis has become situated in a second position of equilibrium, it will float permanently, that is with stability, in that second position. And in general, when a floating solid revolves round a given horizontal axis, and passes through several positions of equilibrium, those of stability and instability are alternate, no position of either species following immediately a position of the same species. In order, therefore, to find what position a solid will assume after it has overset from any situation of unstable equilibrium, it is only necessary to ascertain the angle of inclination from the given situation through which the solid must revolve on the axis of motion, so that the distance GZ (fig. 2.) between the two vertical lines which pass through the centre of gravity of the solid and the centre of gravity of the part immersed may become evanescent. It is necessary in the next place, to determine whether any position of equilibrium originally given is that of stability or instability. This point will be ascertained by having recourse to the general value which has been investigated, for expressing the distance between the two vertical lines GO, ST (fig. 2.); or $GZ = \frac{Ab}{V} - ds$. In the line ER take any point t, and through t draw qtz parallel to GO. As long as $\frac{bA}{V} = ET$ is greater than $ds = ER$, the point Z, and the line of support QZ, will be between the axis and those parts of the solid which are immersed by the inclination,

tion of permanent equilibrium, and is deflected from that position through a small angle, the force of the fluid's pressure causes the solid to revolve round its axis in a direction contrary to the inclination; and if the equilibrium is unstable, the same force acts to increase the said inclination; this latter case corresponds to that of the equilibrium in which the solid is situated after it has revolved through the angle A.

the consequence of which is an equilibrium of stability; and whenever $\frac{bA}{V} = ET$ is less than $ds = ER$, the point q, and the line of support qz, will be on the contrary side of the axis, causing an equilibrium of instability to take place.* The equation, therefore, $GZ = \frac{bA}{V} - ds$, applied to any particular case, will always decide whether the equilibrium in which a solid is placed on the surface of a fluid is stable and permanent, or whether it is only momentary and unstable, provided the value of s, or the sine of the angle of inclination from the given position of equilibrium, be assumed evanescent; since the solid either continues to float permanently, or will overset, according to circumstances which take place while it is inclined from its position of equilibrium through the smallest angle. The application of the condition just mentioned will cause the general expression to assume a form suited to this particular case, which is in the next place to be attended to.

Referring to (fig. 2.), ADHB represents a vertical section of a floating body, passing in a direction perpendicular to the axis of motion; suppose another section to be drawn parallel to the former, and extremely near to it; these two planes will comprehend between them a small portion of the solid; and since according to the conditions of the case, the angle of inclination KGS, or NXB, is evanescent, the sine of this angle (which has been denoted by the letter s) will also become evanescent; and since the space or volume immersed in consequence of the inclination, that is NXP, is equal to the volume elevated above the surface IXW, and the angles NXP, IXW, are vertical; the point of intersection of the lines IN and AB, that is, the point X will bisect the line AB, and the

* Page 4, and page 5.

points P, B, and N, will coincide; on which account the evanescent area NXP will be $= \frac{\overline{XB}^2 \times s}{2} = \frac{\overline{AB}^2 \times s}{8}$; and if z is put to represent a line drawn through the middle of the solid, on a level with the fluid's surface, and parallel to the longer axis, the evanescent portion of the solid intercepted between the two adjacent planes, will be $\frac{\overline{AB}^2 \times s}{8} \times \dot{z}$: the perpendicular distance of the centre of gravity of this evanescent solid from the point X, is $\frac{1}{3}$ AB. But it is required in the present instance to assign the distance from the horizontal line passing through the point X, of the centre of gravity of the entire volume immersed by the inclination, that is, the common centre of gravity of all the evanescent solids $\frac{\overline{AB}^2 \times s\dot{z}}{8}$ corresponding to the entire length z. This distance may be obtained from the known rule of mechanics, which is, by multiplying each evanescent solid, considered as concentered in its centre of gravity, into the distance of that centre from the given line, and dividing the sum of the products by the sum of the solids; the result will be, the distance of the common centre of gravity from the horizontal line passing through the point X parallel to the axis; and since the evanescent solid corresponding to the small lineal increment $\dot{z}$ is $\frac{\overline{AB}^2 \times s \times \dot{z}}{8}$, and the distance of its centre of gravity from the point X $= \frac{2XB}{3}$ or $\frac{AB}{3}$, the product arising from multiplying the solid into the distance of its centre of gravity, from the given horizontal line passing through X, will be $\frac{\overline{AB}^3 \times s \times \dot{z}}{24}$; and the sum of all those products corresponding

to the whole length of the line z will be $\frac{\text{fluent of } \overline{AB}^3 \times s \times \dot{z}}{24}$; and therefore the distance of the common centre of gravity of the volume immersed in consequence of the inclination from the horizontal line passing through the point X, is $\frac{\text{fluent of } \overline{AB}^3 \times s \times \dot{z}}{24 A}$; in like manner the distance of the common centre of gravity of the volume, elevated above the surface by the inclination of the given plane, appears to be $\frac{\text{fluent of } \overline{AB}^3 \times s \times \dot{z}}{24 A}$; and consequently the distance between the two centres of gravity measured on the horizontal line, or bc (fig. 2.) $= \frac{\text{fluent of } \overline{AB}^3 \times s \times \dot{z}}{12 A}$: this value being substituted for b in the equation $GZ = \frac{bA}{V} - ds$, we obtain the following result, *i.e.* $GZ = \frac{\text{fluent of } \overline{AB}^3 \times s \times \dot{z}}{12 V} - ds$, which is a general expression for ascertaining whether a solid, when placed on the surface of a fluid in a given position, will float permanently, or overset, the sine of the angle of inclination or s being assumed evanescent; for, when $\frac{\text{fluent of } \overline{AB}^3 \times s \times \dot{z}}{12 V}$ is greater than ds, the line of support QZ (fig. 2.) will be situated between the axis of motion, and the parts of the solid which are immersed by the inclination, in which case the solid will float permanently; and when $\frac{\text{fluent of } \overline{AB}^3 \times s \times \dot{z}}{12 V}$ is less than ds, the line of support passing through the point z will be on the contrary side of the axis, and according to the preceding determination (page 19) the solid will in this case overset.

Since, when the fluent of $\frac{\overline{AB}^3 s \dot{z}}{12 V}$ (fig. 2.) is greater than ds,

the solid floats permanently; and when ds is greater than $\frac{\text{fluent of } \overline{AB}^3 s\dot{z}}{12V}$, the equilibrium is that of instability; it follows that whenever $\frac{\text{fluent of } \overline{AB}^3 s\dot{z}}{12V} = ds$, by resolving the equation $\frac{\text{fluent of } \overline{AB}^3 \dot{z}}{12V} = d$, one or more limits are obtained (depending on the dimensions and specific gravity of the solid), separating the cases in which the solid floats with stability from those in which the equilibrium is momentary and unstable. The limits here obtained evidently correspond to that species of equilibrium which has been denominated insensible, or the equilibrium of indifference.

When the floating body is of uniform figure and dimensions, respecting the axis of motion, the expression here given for determining the stability or instability of floating will not involve any fluxional quantities, for in this case all the vertical sections which pass through the solid in a direction perpendicular to the axis are equal, and consequently the portions of those sections immersed under the fluid's surface are also equal; if, therefore, the area of any one of these sections immersed under the fluid's surface be denoted by the letter D, the solid contents or volume immersed, corresponding to the length of the line z, will be Dz; wherefore, in the preceding expression $GZ = \frac{\text{fluent of } \overline{AB}^3 \times s \times \dot{z}}{12V} - ds$, we have by substitution $V = Dz$, and since AB is a constant or invariable quantity by the supposition, $\frac{\text{fluent of } AB^3 s\dot{z}}{12Dz} = \frac{AB^3 sz}{12Dz} = \frac{AB^3 \times s}{12D}$: finally, therefore, in the case under consideration, we obtain $GZ = \frac{\overline{AB}^3 \times s}{12D} - ds$.

In the subsequent pages, cases occur in which each of the preceding expressions are employed, not only to ascertain the laws of permanent and unstable equilibrium, but in developing other properties relating to the subject.

EFCD (fig. 3.) represents a vertical section of an oblong solid or parallelopiped, placed on the surface of a fluid IABK, with one of the flat surfaces upward, or the line CE or FD vertical: this solid is moveable round an horizontal axis, which passes through the centre of gravity G, perpendicular to the plane ECDF. Let it be required to determine the limits, depending on the dimensions and specific gravity of the solid, which separate the cases in which the solid will float permanently, from those in which it will overset; through the centre of gravity G draw the line SGL parallel to CE or DF: let the height of the solid $CE = c$; let the base $CD = a$; also let the specific gravity of the solid be to that of the fluid on which it floats in the proportion of n to 1, or as SN to SL; so that when it is placed on the fluid with the line SL vertical, it may sink to the depth SN; let O be the centre of gravity of the part immersed: suppose the solid to be placed on the surface of the fluid with the line SL vertical; then, since SN is the depth to which the solid sinks in the fluid, and SN is to SL as n to 1, it follows that $SN = nc$; and consequently $GO = \frac{c}{2} - \frac{nc}{2}$; the area immersed $ABCD = acn$; wherefore, to ascertain the perpendicular distance between the two verticals which pass through the centres of gravity of the solid and of the part immersed, when the solid is inclined through a very small angle, of which the sine is $= s$ to radius 1, re-

ferring to the general expression * $GZ = \frac{\overline{AB}^3 \times s}{12 D} - ds$, we obtain the following values $AB = a$, $D = acn$, $d = \frac{c - nc}{2}$, and therefore $GZ = \frac{a^3 s}{12 acn} - \frac{s \times \overline{c - nc}}{2}$: by making the distance $GZ = 0$, we obtain an equation expressing the relation of the dimensions and specific gravity of the solid, when the equilibrium becomes insensible, that is, when the centres of gravity of the solid and of the part immersed remain in the same vertical line, however the value of s or the sine of the inclination from the upright position may be altered, provided it is always very small; making, therefore, $\frac{a^3 s}{12 acn} = \frac{s \times \overline{c - cn}}{2}$, we have $6c^2 n^2 - 6c^2 n = - a^2$ and $n^2 - n = - \frac{a^2 s}{6c^2}$, which gives $n = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{a^2}{6c^2}}$, or $n = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{a^2}{6c^2}}$: from whence the following inference is obtained, *i.e.* in all cases whenever $\frac{a^2}{6c^2}$ is less than $\frac{1}{4}$, that is, whenever the height of the solid c bears to the base a a greater proportion than that of $\sqrt{2}$ to $\sqrt{3}$, two values may be assigned to the specific gravity of the solid, each of which will cause it to float in the insensible equilibrium: thus, suppose the height c to be to the base a in the proportion of equality: to ascertain the two limiting specific gravities, by referring to the preceding solution, and making $c = a$, we obtain $n = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{6}}$, or $n = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{6}}$, that is $n = \frac{1}{2} - .28868 = .21132$,

or $n = \frac{1}{2} + .28868 = .78868$.

* When the angle KGS in fig. 2. is evanescent, the line GZ vanishes: this being the case represented by fig. 3, the point Z coincides with the point G.

From the equation $GZ = \frac{a^3 s}{12\,acn} - \frac{s \times \overline{c - cn}}{2}$ it is inferred, that when the specific gravity of the solid is of very small value in respect to that of the fluid, because $\frac{a^3 s}{12\,acn}$ must in this case be necessarily greater than $\frac{s \times \overline{c - cn}}{2}$ the solid will float permanently with the line SL vertical, that is, with the flat surface EF parallel to the horizon. Secondly, the specific gravity .21183 causing the solid to float in the insensible equilibrium, is the limit at which the solid ceases to float with stability; if therefore the specific gravity is increased beyond .21183, and the solid is placed on the fluid with the flat surface upward, the equilibrium thus formed will be that of instability, from which the solid will be deflected into some other position in which the equilibrium is permanent. While the specific gravity is augmented from .211 to .788, the instability increases at first, but admits of a maximum, which is found by putting the least increment of the quantity $\frac{a^3 s}{12\,acn} - \frac{s \times \overline{c - cn}}{2} = 0$, considering n as a variable quantity, and making $a = c$; in which case n appears to be equal to $\frac{1}{\sqrt{6}}$. If the value of the specific gravity is increased beyond $\frac{1}{\sqrt{6}}$, the instability becomes less, and at last vanishes when the specific gravity is at its second limit $= .78868$: whatever value is given to the specific gravity between .78868 and 1, the solid will float permanently with the line SL vertical, or with its flat surface horizontal.

These cases arise from assuming the height of the parallelopiped SL, in a greater proportion to its base CD than that of

$\sqrt{2}$ to $\sqrt{3}$; and from the same solution it appears, that if the height bears a less proportion to the base than that of $\sqrt{2}$ to $\sqrt{3}$, no value can be given to the specific gravity, which will cause the stability to vanish, because the quantity $\sqrt{\frac{1}{4} - \frac{a^2}{6c^2}}$ becomes impossible; in which case the solid placed with the surface EF horizontal, must in all cases continue to float permanently in that position, whatever may be the specific gravity, always supposed to be less than that of the fluid.

(Fig. 4.) Similar determinations may be obtained from the same theorem respecting the equilibrium of the solid, when placed on a fluid with a plane angle upward, that is, with the diagonal line EGC vertical. Let EDCF represent a vertical section of a square parallelopiped floating on the surface of a fluid IABK: making the side DC $= a$, the line GC $= \frac{a}{\sqrt{2}}$, suppose that the specific gravity of the solid is to the specific gravity of the fluid as n to 1, and that the solid sinks in the fluid to the depth HC; let G be the centre of gravity of the solid, and O the centre of gravity of the part immersed; then the area ABC is to the area DEFC as n to 1; wherefore the space ABC $= \overline{HB}^2 = a^2 n$, and HB $=$ HC $= a \times \sqrt{n}$; AB $= 2a\sqrt{n}$; OC $= \frac{2a\sqrt{n}}{3}$ and GO $= \frac{a}{\sqrt{2}} - \frac{2a\sqrt{n}}{3} = \frac{a \times 3 - \sqrt{8 \times n}}{\sqrt{2} \times 3}$.

Referring to the quantity expressing the perpendicular distance between the two vertical lines passing through the centre of gravity of the solid, and the centre of gravity of the part immersed, when the angles of inclination from the

position of equilibrium, are very small, that is, $GZ = \frac{\overline{AB}^3 \times s}{12 D} - ds$, and applying this equation to the case under considera-tion, we obtain the following values; $AB^3 = 8a^3 \times n^{\frac{3}{2}}$; $D = a^2 n$; $d = \frac{a \times \overline{3 - \sqrt{8n}}}{\sqrt{2} \times 3}$: making therefore $\frac{AB^3 s}{12 D} = ds$, in order to obtain the limit, separating the cases of stability and instability of floating; or, which is the same thing, making $\frac{8a^3 n^{\frac{3}{2}}}{12a^2 n} = \frac{a \times \overline{3 - \sqrt{8n}}}{\sqrt{2} \times 3}$, the following equation arises, $\frac{2\sqrt{n}}{3} = \frac{3 - \sqrt{8n}}{\sqrt{2} \times 3}$, or $n = \frac{9}{32} = .28123 =$ the specific gravity, which will cause the solid to float in the insensible equilibrium, and is therefore the limit separating the specific gravities which cause the solid to float with stability from those which produce the equilibrium of instability. It is collected from the general equation $GZ = \frac{\overline{AB}^3 s}{12 D} - ds$, or $GZ = \frac{8a^3 n^{\frac{3}{2}} s}{12a^2 n} - \frac{a \times \overline{3 - \sqrt{8n}}}{\sqrt{2} \times 3}$; that when the specific gravity (n) is evanescent or very small, the solid will overset when placed on the fluid with an angle upward, because in this case the quantity $\frac{8a n^{\frac{3}{2}} s}{12a^2 n}$ must necessa-rily be less than $\frac{a \times \overline{3 - \sqrt{8n}}}{\sqrt{2} \times 3}$, or ds. When the specific gravity of the solid is to that of the fluid in the proportion of 9 to 32, the solid floats in the insensible equilibrium; if therefore the specific gravity of the solid should be to that of the fluid in a less proportion than that of 9 to 32, the solid will overset; but if the specific gravity of the solid exceeds that limit when placed on the fluid with the angle upward, or diagonal line EC vertical, it will float permanently in that position.

Respecting this determination it seems remarkable, that there should be only one value of specific gravity, as a limit between the stability and instability of floating; whereas there were two specific gravities, each of which was a limit in the case when the solid was placed on the fluid with a flat surface upward. This difficulty admits of very satisfactory explanation; when the flat surface is placed upward, the conditions on which the solution is founded are not at all altered, to whatever depth the solid may sink: but in the present case, when the solid is placed on the fluid with a plane angle upward, the conditions on which the solution has been investigated imply, that as the specific gravity is increased, the section of the solid formed by the fluid's surface shall continually increase also; and on that ground the result justly gives one limit only between the stability and instability of floating; but since in reality the section of the solid by the fluid's surface increases only until the specific gravity becomes one half of that of the fluid, and afterwards decreases, it is evident, that if there should be another limit corresponding to the case when the specific gravity is greater than one-half, it must be discovered by a separate investigation. Let, therefore, the square parallelopiped EDCF (fig. 5.) of which the specific gravity is greater than $\frac{1}{2}$, that of the fluid being 1, be placed on the fluid with the diagonal line EC vertical: IABK represents the surface of the fluid, and HC the depth to which the solid sinks; G is the centre of gravity of the solid, and O the centre of gravity of the part immersed. If one of the sides DE is made $= a$, and the specific gravity put $= n$, then the area ABDCFA $= a^2 n$; and the area EAB $= a^2 - a^2 n = \overline{EH}^2$; wherefore EH $= a \times \sqrt{1 - n} =$ AH; AB $= 2a \times$

$\sqrt{1-n}$; and GH $= a \times \overline{\frac{1}{\sqrt{2}} - \sqrt{1-n}}$: let P represent the centre of gravity of the area AEB; then by the properties of the centre of gravity the following equation arises:
GH × area EDCF = area ABDCFA × OH — area AEB × HP, that is

$a^3 \times \overline{\frac{1}{\sqrt{2}} - \sqrt{1-n}} = a^2 n \times \mathrm{OH} - a^3 \times \frac{\overline{1-n}^{\frac{3}{2}}}{3}$; and consequently HO $= \frac{a \times \overline{3 - \sqrt{18} \times \sqrt{1-n}} + a \times \sqrt{2} \times \overline{1-n}^{\frac{3}{2}}}{\sqrt{18} \times n}$; from which quantity taking away the line HG $= a \times \overline{\frac{1}{\sqrt{2}} - \sqrt{1-n}} = \frac{3n - \sqrt{18n^2} \times \sqrt{1-n}}{\sqrt{18} \times n}$, there will remain the line GO $=$

$\frac{a \times \overline{3 - 3n - \sqrt{18} \times \overline{1-n}^{\frac{3}{2}} + \sqrt{2} \times \overline{1-n}^{\frac{3}{2}}}}{\sqrt{18} \times n}$.

Referring to the general expression, namely $\frac{\overline{AB}^3 s}{12D} - ds$, we obtain in the present case $\overline{AB}^3 = 8a^3 \times \overline{1-n}^{\frac{3}{2}}$, $D = a^2 n$, GO $= d = \frac{a \times \overline{3 - 3n - \sqrt{18} \times \overline{1-n}^{\frac{3}{2}} + \sqrt{2} \times \overline{1-n}^{\frac{3}{2}}}}{\sqrt{18} \times n}$; wherefore $\frac{\overline{AB}^3}{12D} - d = \frac{8a^3 \times \overline{1-n}^{\frac{3}{2}}}{12a^2 n} - \frac{a \times \overline{3 - 3n - \sqrt{18} \times \overline{1-n}^{\frac{3}{2}} + \sqrt{2} \times \overline{1-n}|^{\frac{3}{2}}}}{\sqrt{18} \times n}$; which quantity being put equal to o, in order to obtain the limit, and the whole being multiplied by $\frac{3n \times \sqrt{2}}{\overline{1-n} \times a}$, will give $2\sqrt{2} \times \sqrt{1-n} = 3 - 3\sqrt{2} \times \sqrt{1-n} + \sqrt{2} \times \sqrt{1-n}$, or $\sqrt{1-n} = \frac{3}{4\sqrt{2}}$; wherefore $1 - n = \frac{9}{32}$ or $n = \frac{23}{32}$, the limit required.

By the preceding determinations of the four limiting values of

the specific gravity, *i.e.* $\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{6}}$, $\frac{9}{32}$, $\frac{23}{32}$, & $\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{6}}$: or .211, .281, .718, & .789, we find that if the specific gravity is less than .211, the square parallelopiped, when placed on the surface of the fluid with a flat surface upward and horizontal, floats permanently in that position, but oversets if the specific gravity is greater than .211, and less than .789. We observe also, that when the solid is placed on the fluid with an angle upward, if the specific gravity is less than .281 it oversets; if greater than .281 and less than .718, the solid floats permanently with the angle upward; but if the specific gravity exceeds .718, the solid oversets when placed on the fluid with an angle upward.

It is therefore evident at what depth of floating, depending on the specific gravity, the solid when placed on the fluid in the positions which have been described, begins or ceases to float with stability. But a material inquiry remains to be considered, which is, to ascertain in what position a square parallelopiped will dispose itself, in respect to the fluid's surface, when the specific gravity is of any intermediate values between the limits which have been determined. To resolve this question the preceding results are evidently inadequate, since from these we only know in what cases, depending on the values of the specific gravity, the solid when placed on the fluid either with a flat surface or an angle upward will float permanently; and in what cases it will overset. Suppose the latter event to take place, and that the solid, having been placed on the fluid in a position of unstable equilibrium, oversets or changes its position by revolving on its axis. To ascertain what position the solid so circumstanced will assume, in which

it will continue permanently to float, we must have recourse to the theorem for expressing the perpendicular distance between the two verticals, which pass through the centres of gravity of the solid and of the part immersed. For by putting this value = o, the resolution of an equation thence arising, will give the sine of the inclination from the position of equilibrium at which these two vertical lines coincide; that is, when the centres of gravity of the solid and of the part immersed are again in the same vertical line: in this case the solid will be situated in a position of equilibrium, which, according to the observations in page 18, must be an equilibrium of stability.

Let EFDC (fig. 6.) represent the vertical section passing through the centre of gravity G of an oblong solid or parallelopiped, the longer axis of which passes through the centre of gravity G in a direction perpendicular to the plane EFCD; LGS is drawn through G parallel to CE or DF; this solid is placed on the surface of a fluid IABK, with the line SGL vertical; and the specific gravity of the solid is such as causes it to sink to the depth under the fluid's surface SN.

The volume immersed under the fluid's surface is the space ACDB, of which the centre of gravity is O; and since the points G and O are situated in the same vertical line, the solid will be in a position of equilibrium, which, according to the present supposition, is assumed to be the equilibrium of instability; the solid will therefore spontaneously overset whenever external support is removed, and will change its position by revolving round an horizontal axis which passes through the centre of gravity in a direction perpendicular to the plane CDFE.

It is required to ascertain through what angle WGS, the solid will be inclined round its axis, when the centres of gravity of the solid and of the part immersed are again in the same vertical line. As in the former cases, this problem will be solved, by referring to the general expression for the distance between the two vertical lines which pass through the centres of gravity of the solid and of the part immersed.

Suppose then the solid to be inclined from its former position of equilibrium in an angle WGS, so as to become transferred from the position ECDF into the position YWHV; the part immersed will now be ZHVR; the line AB will also be transferred to PQ, and the space QXR, which was before above the fluid's surface, will now be immersed under it; and the space PXZ, which was before under the surface, will now be above it. Bisect the lines PZ, QR, in m and n, and join mX, nX; and take X$a = \frac{2}{3}$ of Xm and X$d = \frac{2}{3}$ of Xn; so shall a and d be the centres of gravity of the triangles PXZ, QXR, respectively; draw the lines ab, cd, perpendicular to the horizontal line AB. Referring to the quantity expressing the distance between the vertical lines which pass through the centres of gravity of the solid and of the part immersed, namely, $\frac{bA}{V} - ds$, there will be applicable to the present case, the space QXR $=$ A; the space ZHVR or ACDB $=$ V; $bc = b$; OG $= d$; the sine of the angle of inclination or WGO $= s$: let t be the tangent of the same angle to radius $= 1$; then, since the triangles ZXP, QXR, are similar, and the areas are equal by the supposition, the sides of the two triangles will be respectively equal; that is, QX will be equal to XP; ZP equal to QR; and ZX to XR. Let the height of the solid SL $= c$,

and the specific gravity $= n$ when that of the fluid is equal to 1, also make VW or XQ $= a$; then QR $= at$, and $Qn = \frac{at}{2}$; $Xn = \sqrt{a^2 + \frac{t^2 a^2}{4}}$, or $Xn = \frac{a}{2} \times \sqrt{4 + t^2}$.

To find the sine of the angle nXR, make the following proportion. As Rn or Qn $\left(\frac{ta}{2}\right)$: Xn $\left(\frac{a}{2} \times \sqrt{4+t^2}\right)$: : sine nXR : sine XRn: wherefore sine $n\mathrm{XR} = \frac{\text{sine } n\mathrm{RX} \times t}{\sqrt{4+t^2}}$; or because sine $n\mathrm{RX} = \frac{1}{\sqrt{1+t^2}}$, sine $n\mathrm{XR} = \frac{t}{\sqrt{4+t^2} \times \sqrt{1+t^2}}$; cos. $n\mathrm{XR} = \frac{2+t^2}{\sqrt{4+t^2} \times \sqrt{1+t^2}}$: and since $\mathrm{X}d = \frac{2}{3} \times \mathrm{X}n = \frac{a \times \sqrt{4+t^2}}{3}$, it follows that $\mathrm{X}c = \frac{a \times \sqrt{4+t^2} \times \overline{2+t^2}}{3n\sqrt{4+t^2} \times \sqrt{1+t^2}} = \frac{a}{3} \times \frac{2+t^2}{\sqrt{1+t^2}}$; and since the triangles XPZ, XQR, as also the triangles ZXm, RXn, are similar and equal, the line Xb = Xc; and consequently $bc = 2\mathrm{X}c = \frac{2a \times \overline{2+t^2}}{3 \times \sqrt{1+t^2}}$; which quantity $= b$ in the general value $\frac{b\mathrm{A}}{\mathrm{V}} - ds$. And since the specific gravity of the solid is $= n$, the height SL $= c$, and the base CD $= 2a$, the immersed part or ACDB $= 2acn$, which in the general expression is denoted by V; and the volume QXR $= \frac{a^2 t}{2}$ is denoted by the letter A in such general value.

Substituting, therefore, in the expression $\frac{b\mathrm{A}}{\mathrm{V}} - ds$, $\frac{2a \times \overline{2+t^2}}{3 \times \sqrt{1+t^2}}$ for b; $\frac{a^2 t}{2}$ for A; and $2acn$ for V; the distance between the vertical lines passing through the centres of gravity of the solid and the centre of gravity of the part immersed, appears

to be $\frac{2a \times \overline{2 + t^2}}{3 \times \sqrt{1 + t^2}} \times \frac{a^2 t}{2 \times 2acn} - ds$, or $\frac{a^2 t \times \overline{2 + t^2}}{6cn \times \sqrt{1 + t^2}} - ds$; or since $d = \frac{c - cn}{2}$, the said distance $= \frac{a^2 t \times \overline{2 + t^2}}{6cn \times \sqrt{1 + t^2}} - \frac{\overline{c - cn} \times s}{2}$; or by substituting for t^2 its value $\frac{s^2}{1 - s^2}$, the distance $= \frac{a^2 s \times \overline{2 - s^2}}{6cn \times \overline{1 - s^2}} - \frac{\overline{c - cn} \times s}{2}$; in which expression a denotes half the breadth PQ; but as it may be more convenient to represent the whole breadth AB or PQ by the letter a, the expression will in this case be $= \frac{a^2 s \times \overline{2 - s^2}}{24cn \times \overline{1 - s^2}} - \frac{\overline{c - cn} \times s}{2}$; which quantity being put $= 0$, we obtain $s^2 = \frac{2a^2 - 12c^2 n + 12c^2 n^2}{12c^2 n^2 - 12c^2 n + a^2}$, or $s^2 = \frac{12c^2 n - 12c^2 n^2 - 2a^2}{12c^2 n - 12c^2 n^2 - a^2}$. From this equation the angle of inclination from the original position of equilibrium may be found, from having given the specific gravity; or conversely, the specific gravity may be found from having given the angle of inclination through which the solid must revolve, so as to be situated in a second position of equilibrium. As the instances given to illustrate the propositions already investigated have been adapted to the case of a square parallelopiped, the present result may be exemplified on the same supposition. Assuming then the height of the solid to be equal to the base, a will become $= c$ in the preceding expression, and consequently $s^2 = \frac{12n - 12n^2 - 2}{12n - 12n^2 - 1}$.

We have seen in a foregoing proposition, that if the specific gravity of this square solid should be greater than .211 so as not to exceed .789, the solid placed on the fluid with a flat surface upward, would be situated in an equilibrium of instability, and consequently must change its position by revolving on

its axis till it settles in some other position wherein the equilibrium is permanent.

From the present proposition we shall be enabled to ascertain what that position is. Thus, let the specific gravity $n = .24$, which is between the limits .211 and .789; and will consequently place the solid with a flat surface upward and horizontal, in an equilibrium of instability. By referring to the equation $s^2 = \frac{12n - 12n^2 - 2}{12n - 12n^2 - 1}$, and substituting .24 for n, we find that $s^2 = \frac{12n - 12n^2 - 2}{12n - 12n^2 - 1} = \frac{.1888}{1.1888}$; and $s =$ the sine of $23° 29'$. From this calculation it appears, that the solid after having overset from its position of unstable equilibrium, with the flat surface upward and horizontal, and having revolved through an angle of $23° 29'$, will settle in a position of permanent equilibrium at that angular distance from its original situation; for by the solution, when the solid has revolved through that angle, the centres of gravity of the solid and of the part immersed are again situated in the same vertical line, and consequently the solid is then situated in a position of equilibrium, which must be the equilibrium of stability, because the original position from which the solid inclined, was that of instability; and it has been observed previously, that when a solid changes its position by revolving on an axis on the surface of a fluid, any position of equilibrium is always succeeded by a position of equilibrium which is of a contrary description.

If the angle of inclination from the upright position with a flat surface horizontal should be given, the specific gravity of the solid may be inferred from the preceding equation, which will cause the solid to float in a position of equilibrium at

that given angle of inclination; for by solving the equation $s^2 = \frac{12n - 12n^2 - 2}{12n - 12n^2 - 1}$ we obtain $n = \frac{1}{2} \pm \sqrt{\frac{1 - 2s^2}{12 - 12s^2}}$. Thus, if it should be required to ascertain the specific gravity which will cause the solid to float *in equilibrio* at the angular distance of 23° 29′ from the upright, we have $\sqrt{\frac{1 - 2s^2}{12 - 12s^2}} =$ 0.26000, and the specific gravity required, that is, $n = .5 +$ $.26 = .76$, or $n = .5 - .26 = .24$. Thus we find from this calculation that there are two specific gravities which will cause the solid to float in a position of equilibrium at the same angular distance 23° 29′ from the original situation with a flat surface horizontal; a conclusion which it is easy to verify by substituting .76 for n in the equation $\frac{12n - 12n^2 - 2}{12n - 12n^2 - 1} = s^2$: the result is that $s^2 = \frac{.1888}{1.1888}$, the same as in the former instance, when n was assumed $= .24$.

In the application of analytical investigation to the solution of problems, it is always necessary to keep distinctly in view the conditions on which the investigation has been founded; for however correct the solution may otherwise have been, any inadvertence in this respect will unavoidably lead to error and inconsistency. The investigation by which the floating position of the solid is determined after it has changed its position from an equilibrium of instability, when one of the flat surfaces was parallel to the horizon, has proceeded on a supposition that the surface of the fluid intersects the parallel surfaces YH, WV, (fig. 6.) in the points R and Z; but if the two surfaces intersected by the fluid should be the inclined sides HV, VW, or in other words, if the point of intersection Z should be situated between H and V, neither the geometrical construction

nor the analytical investigation depending on it, can be applied, so as to ascertain the required position of equilibrium, a solution altogether different being required to determine the position in which a solid under these conditions will float permanently. It is, however, certain, that as long as the point of intersection Z is not lower than the point of the base H, the preceding solution will be applicable: it will be therefore material to find both the angle of inclination from the original position of unstable equilibrium, and the specific gravity of the solid when it floats permanently, with this condition annexed, *i. e.* that the surface of the fluid shall pass through one of the extremities of the base: the result of this solution will form a limiting value both of the angle of inclination and of the specific gravity, beyond which the preceding investigation not being applicable, another solution is required.

Let AECD (Tab. IV. fig. 7.) represent a vertical section of the square parallelopiped which rests permanently on the surface of the fluid IKDH, passing through the extremity of the base D. It is required to find the angle of inclination KDC from a position of equilibrium with a flat surface horizontal, and the specific gravity of the solid, when it floats in a state of equilibrium. Let the tangent of the required angle KDC be to radius as t to 1, and put CD $= a$; let the specific gravity of the solid be to that of the fluid as n to 1. Then KC $= at$, and the area KCD $= \frac{a^2 t}{2}$: and because as the area KCD is to the area AECD, so is n to 1, it follows that $n = \frac{t}{2}$; and since by the preceding investigation* $s^2 = \frac{12n - 12n^3 - 2}{12n - 12n^2 - 1}$, where s represents the sine of the angle of inclination from the

* Page 34.

upright position, which is the angle KDC in the present case; substituting for n its value $\frac{t}{2}$, the equation will now become $s^2 = \frac{6t - 3t^2 - 2}{6t - 3t^2 - 1}$, or because $s^2 = \frac{t^2}{1 + t^2}$, $\frac{t^2}{1 + t^2} = \frac{6t - 3t^2 - 2}{6t - 3t^2 - 1}$, or $6t^3 - 3t^4 - t^2 = 6t - 3t^2 - 2 + 6t^3 - 3t^4 - 2t^2$, or $4t^2 = 6t - 2$; which equation being resolved, gives $t = \frac{3}{4} \pm \frac{1}{4}$, that is, $t = \frac{1}{2}$ or $t = 1$. By this solution it appears, that there are two angles at which the solid may be inclined from its upright position of unstable equilibrium with the flat surface upward, so as to rest permanently on the surface of the fluid, when that surface passes through one extremity of the base: 1st, when the angle of inclination is KDC $= 26° \; 33', 51'',4$, or about $26° \; 34'$, of which the tangent is to radius as 1 to 2; and secondly, (fig. 8.) when the angle of inclination KDC $= 45°$, of which the tangent is equal to the radius. When the solid floats permanently on the fluid at the angle of inclination KDC $= 26° \; 34'$ from the upright position, the part immersed, or KCD, is to the whole volume ABCD as 1 to 4; and therefore the specific gravity of the solid is to that of the fluid as 1 to 4, or resuming the former notation applied to the present case, the specific gravity of the solid or $n = \frac{1}{4}$, when that of the fluid is $= 1$. That the position of equilibrium here determined is that of stability, appears from attending to the limiting value of the specific gravity, determined in page 24, where it is shewn that when the square parallelopiped is placed on the surface of a fluid with one of the flat surfaces horizontal, and the specific gravity of the solid is greater than .211, so as not to exceed .789, the equilibrium will be that of instability, and consequently the solid will overset. It has

been just shewn, that after the body has revolved through an angle of $26° 34'$ it will be again in a position of equilibrium, which must therefore be the equilibrium of stability. Similar consequences follow from supposing the specific gravity $= \frac{1}{2}$; in this case if the solid is placed on the fluid with a flat surface upward, the equilibrium will be that of instability; and it appears from the preceding solution, that after revolving through an angle of $45°$, (fig. 8.) it will again be in a position of equilibrium, which therefore will be stable and permanent. By a similar investigation, the angle of inclination ABK (fig. 9.) from the original position of equilibrium may be found when the solid floats permanently, and the fluid's surface intersects one of the extremities of the upper side of the square AB: for the notation remaining, by putting the tangent of the angle of inclination $ABK = t$, the area $ABK = \frac{a^2 t}{2}$, area $KCDB = \frac{2a^2 - a^2 t}{2}$, wherefore the specific gravity or $n = \frac{2 - t}{2}$; which quantity being substituted for n in the equation $\frac{t^2}{1 + t^2}$* $= \frac{12n - 12n^2 - 2}{12n - 12n^2 - 1}$, there will arise the equation $\frac{t^2}{1 + t^2} = \frac{6t - 3t^2 - 2}{6t - 3t^2 - 1}$, exactly the same as in the former case; and by solving this equation it appears that $t = \frac{3}{4} \pm \frac{1}{4}$, and consequently the specific gravity of the solid, or $n = \frac{2 - t}{2} = \frac{3}{4}$ or $n = \frac{1}{2}$.

The only inquiry remaining to complete the investigation respecting the floating positions of the square parallelopiped, is to ascertain in what position the solid will float permanently

* Because s being the sine, and t being the tangent of the angle ABK, it follows that $s^2 = \frac{t^2}{1 + t^2}$.

with a plane angle obliquely upward, when the specific gravity is between the limits $\frac{8}{32}$ and $\frac{9}{32}$, or between the limits $\frac{23}{32}$ and $\frac{24}{32}$. It has been seen in a former investigation, that if the solid is placed on the fluid with an angle upward, and the specific gravity is $\frac{9}{32}$, it will just begin to float with stability, and ceases to float with stability when the specific gravity exceeds $\frac{23}{32}$. When the specific gravity is $\frac{1}{4} = \frac{8}{32}$ or $\frac{3}{4} = \frac{23}{32}$, it floats permanently with the surface of the fluid coincident with an extremity of one of the sides: if, therefore, the specific gravity is between the limits $\frac{8}{32}$ and $\frac{9}{32}$, or between $\frac{23}{32}$ and $\frac{24}{32}$, the solid will float permanently, with the diagonal line inclined to the vertical. This angle may be determined by finding an equation which expresses the relation between the given specific gravity and the sine or tangent of the required angle to radius $= 1$. Let a square parallelopiped IVCF (fig. 10.) float with an angle obliquely upward, so that the diagonal line shall make an angle with the vertical; suppose that angle to be OGT, the line GT being perpendicular to the horizon; let the surface of the fluid coincide with the line DE perpendicular to GT; take CB a mean proportional between EC and CD, and draw BA parallel to GV, intersecting the line GC in H; so shall CH be the depth to which the solid sinks in the fluid when the diagonal line CI is vertical, and consequently the area BXE is equal to the area XDA; take CO $= \frac{2}{3}$ CH; O will be the centre of gravity of the volume ABC; bisect EB in K, and AD in B; draw XR and XK; and take XM $= \frac{2}{3}$ of XR, and HL $= \frac{2}{3}$ of XK; M will be the centre of gravity of the

triangle XAD, and L will be the centre of gravity of the triangle BXE; through the points M, L, draw the lines MP, QL, perpendicular to the horizontal line DE; make PQ $= b$, the sine of BXE $= s$; the tangent of BEX $= t$ to radius $= 1$, and let EC $= a$.

Then CD $= ta$; and CB $= \sqrt{ta^2}$; CH $= \sqrt{\frac{ta^2}{2}}$; CO $= \frac{2}{3} \times$ CH $= \sqrt{\frac{2ta^2}{9}}$; the area ABC $= \overline{CH}^2 = \frac{ta^2}{2}$; put the area BXE $= u$; then to find the distance OT, the following proportion is to be made; as the area CDE or ABC is to the area BXE : : so is PQ* to OT; or as $\frac{ta^2}{2} : u :: b :$ OT $= \frac{2bu}{ta^2}$; and OG $= \frac{2bu}{ta^2 s}$; and since CO $= \sqrt{\frac{2ta}{9}}$, it follows that CG $= \frac{2bu}{ta^2 s} + \sqrt{\frac{2ta^2}{9}}$; and therefore CV $= \frac{\sqrt{8} \times bu}{ta^2 s} + \sqrt{\frac{4ta^2}{9}} = \frac{\sqrt{72} \times bu + \sqrt{4t^3 a^6 s^2}}{3ta^2 s}$; and the specific gravity being $= n$, $\sqrt{n} = \frac{\text{CH}}{\text{CV}} = \sqrt{\frac{ta^2}{2}} \times \frac{3ta^2 s}{\sqrt{72} \times bu + \sqrt{4t^3 a^6 s^2}} = \frac{3t^{\frac{3}{2}} a^3 s}{12bu + 2\sqrt{2}\, t^{\frac{3}{2}} a^3 s}$: thus, if the angle at which the diagonal line IC is inclined to the vertical line TN or OGT $=$ BXE should be $15°$, the angle XEC $= 30°$; wherefore in the preceding expression, $t =$ tangent $30°$ to radius 1; $s =$ sine $15°$; if CE or a is assumed $= 1$, on making the proper trigonometrical computations, the area BXE $= u = .039395$, and PQ $= b = 0.73089$; from substituting these quantities for their values in the equation $\sqrt{n} = \frac{3t^{\frac{3}{2}} a^3 s}{12bu + 2\sqrt{2t^{\frac{3}{2}}}\, a^3 s}$, it appears that $\sqrt{n} = \frac{.34063}{.34552 + .32114} = 0.51094$, and $n = 0.261$ the specific gravity which causes the solid to float on the fluid in a position of equilibrium with a

* Page 14.

diagonal line obliquely upward, being inclined to the vertical at an angle of $15°$; the equilibrium is that of stability, because when the diagonal is vertical, the solid floats in a position of unstable equilibrium, the specific gravity 0.261 being less than $\frac{9}{32}$ or .281, the limiting value which separates the cases of permanent and unstable equilibrium when the solid is placed on the fluid with a diagonal line vertical.

It is curious to observe the conclusions which arise in the extreme case when the angle of inclination from the vertical is assumed $= 0$; and consequently the angle XEC $= 45°$; for in this case CB $=$ CE $= a$; $t = 1$; and BH $= \frac{a}{\sqrt{2}}$; therefore u or the area BXE * $= \frac{\overline{BH}^2 \times s}{2} = \frac{sa^2}{4}$; and since $b = PQ = \frac{4a}{3\sqrt{2}}$, it follows that $bu = \frac{a^3 s}{3\sqrt{2}}$; and $12bu = \frac{4a^3 s}{\sqrt{2}} = \sqrt{2}a^3 s$; which quantities being substituted for their values, the equation $\sqrt{n} = \frac{3t^{\frac{3}{2}} a^3 s}{12bu + 2\sqrt{2t^{\frac{3}{2}} a^3 s}}$ will become $\sqrt{n} = \frac{3a^3 s}{2\sqrt{2a^3 s} + 2\sqrt{2a^3 s}} = \frac{3}{4\sqrt{2}}$; and therefore $n = \frac{9}{32}$, agreeing† precisely with the specific gravity inferred by a different method from the same data.

The equation $\sqrt{n} = \frac{3t^{\frac{3}{2}} s}{2\sqrt{2t^{\frac{3}{2}} s} + 12bu}$ (the line CE $= a$ being assumed $= 1$) expresses the relation between the specific gravity of the solid and the fraction representing the sine of the angle of inclination from the upright position: if, therefore, that

* Because the point of intersection X coincides with H when the angle BXE vanishes.

† Page 27.

angle is given, the specific gravity will be known. If it should be required to find the sine of the angle of inclination from having given the specific gravity, it is evident from the nature of the equation, that such determination would require analytical operations extremely complex and troublesome, which may be avoided by having recourse to well known methods of approximation. By assuming the quantities s and t by estimation, let the value of $\sqrt{n}$ be calculated from the equation, which being compared with the given value of $\sqrt{n}$, the difference will be the error arising from the error in the assumed values of s and t, which are therefore to be corrected, and the operation repeated until the value of $\sqrt{n}$, deduced from calculation, coincides with its true value; from which method of proceeding, the angle of inclination from the original position of equilibrium will be known.

This solution is evidently applicable to all cases in which the specific gravity of the solid is between the limits $\frac{8}{32}$ and $\frac{9}{32}$, and by an investigation entirely similar, an equation is deduced expressing the relation of the specific gravity of the solid and the sine or tangent of the angle of inclination from the perpendicular, when the specific gravity of the solid is between $\frac{23}{32}$ and $\frac{24}{32}$; in which case the solid will float permanently with the diagonal line IC obliquely upward, being inclined to the vertical at some angle between the limits 0 and $18° 26' 8'',6$.

These determinations comprehend all the positions in which a square parallelopiped can be placed on the surface of a fluid in a position of equilibrium, provided the solid is moveable only round one axis, namely, that which passes through the centre

of gravity perpendicular to the planes of the square sections; and this condition is insured by making the axis of sufficient length; for instance, if it is two or three times longer than one of the sides, the solid will not spontaneously revolve on any other axis. When the axis is diminished considerably, it is certain the body will be spontaneously moveable round some other axis; but it is unnecessary to enter into a detail of multiplied instances, since the exposition of principles is the material object in disquisitions of this kind.

The various positions which the square parallelopiped assumes when floating freely on a fluid's surface depending on its specific gravity, are brought under one point of view in the following abstract, the line IK denoting the surface of the fluid in the figures from fig. 11 to fig. 24.

If the specific gravity of the solid should be between the limits o and $\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{6}}$, (fig. 11, 12, and 13.) that is, between o and 0.211, the solid floats permanently on the fluid with a flat surface upward, and parallel to the horizon.

If the specific gravity is between the limits .211 and .25 (fig. 13, 14, and 15.), the solid floats permanently with a flat surface upward, but inclined to the horizon at sundry angles of which the limits are 0°, corresponding to the specific gravity .211 and 26° 34′, corresponding to the specific gravity .25.

If the specific gravity is between the limits $.25 = \frac{8}{32}$ and $\frac{9}{32}$, (Tab. IV. and Tab. V. fig. 15, 16, 17.) the solid floats with one angle only immersed under the fluid's surface, the diagonal line being inclined to the vertical at various angles depending on the specific gravity, the limits of which angles are 18° 26′, cor-

responding to the specific gravity $.25 = \frac{8}{32}$, and 0, corresponding to the specific gravity $\frac{9}{32}$.

When the specific gravity is increased beyond $\frac{9}{32}$, (fig. 17, 18.) the solid floats permanently with a diagonal line vertical, till the specific gravity becomes $= \frac{23}{32}$.

If the specific gravity is of any magnitude between $\frac{23}{32}$ and $\frac{24}{32}$ the solid floats with the diagonal line inclined to the vertical at sundry angles depending on the specific gravity, (fig. 18, 19, 20.) the limits of which angles are 0, corresponding to the specific gravity $\frac{23}{32}$, and 18° 26′ corresponding to the specific gravity $\frac{24}{32}$, three angles of the solid being immersed under the fluid's surface.

If the specific gravity is between the limits $\frac{24}{32}$ and .789, (fig. 20, 21, 22.) the solid floats with a flat surface upward, and inclined to the horizon at sundry angles depending on the specific gravity, the limits of which angles are 26° 34′ corresponding to the specific gravity $\frac{24}{32}$ or .75, and 0 corresponding to the specific gravity .789.

When the specific gravity is of any magnitude between .789 and 1, the solid floats permanently with a flat surface parallel to the horizon.

From these determinations we also collect that while the solid in question, floating on the fluid's surface, revolves round its longer axis through 360°, it passes through either 16 or 8 positions of equilibrium. If the specific gravity should be between the limits .211 and .281, or between the limits .719 and

.789, the number of those positions will be sixteen; of which eight will be positions of permanent, and the remaining eight positions of unstable equilibrium; these different species of equilibrium succeeding each other alternately while the solid revolves round its axis. If the specific gravity should be of any value not included within these limits, the solid in revolving through 360° will pass through 8 positions of equilibrium only; of which four are positions of permanent, and four of unstable equilibrium.

In the investigations which have preceded, the solid is supposed to be of uniform figure in respect to the axis of motion, so as to make all the vertical sections drawn perpendicular to the axis equal. But when the floating body is of such a form that the sections drawn through it perpendicular to the axis at various points thereof are unequal, a different process, depending however on the same principles, will be necessary; both for determining whether the solid will float permanently or overset, and for ascertaining the several positions in which it will float on the surface of a fluid.

Let EFCD (fig. 23.) represent a cylinder* placed on the surface of a fluid with the axis NP vertical. Suppose the specific gravity to be such as causes the solid to sink to the depth QP; let it be required to determine in what cases, depending on the dimensions and specific gravity of the cylinder, it will float permanently in that position, and in what cases it will overset. Put the radius QA $= r$; the specific gravity of the solid $= n$, that of

* In this and the following propositions, the plane surfaces which terminate the solid are always understood to be perpendicular to the axis.

the fluid being $= 1$; let the centre of gravity be G; the centre of gravity of the immersed part $=$ O; GO $= d$; let AIBHSA represent a circular section of the cylinder coincident with the fluid's surface; draw any diameter IS; and a diameter AB perpendicular to IS; let the axis passing through the centre of gravity round which the cylinder is moveable be parallel to IS; through any point W of the diameter IS draw the ordinate KW perpendicular to IS, and produce KW till it intersects the circle in the point H; make QW $= z$; NP $= l$; $\pi = 3.14159$. It appears from page 21 that the solid will float permanently in the given position of equilibrium with the axis vertical, when the fluent of $\frac{\overline{KH}^3 \times \dot{z}}{12V}$ is greater than d, the letter V signifying the volume immersed under the fluid's surface; it is also shewn in page 21, that if d is greater than $\frac{\text{fluent of } \overline{KH}^3 \times \dot{z}}{12V}$, the equilibrium will be unstable; when the fluent of $\frac{\overline{KH}^3 \times \dot{z}}{12V} = d$, the equilibrium will be the limit separating the cases in which the solid floats with stability from those in which it is momentary and unstable. To ascertain the limit in the present case it is necessary to find the fluent of $\frac{\overline{KH}^3 \times \dot{z}}{12V}$. Since QS $= r$, and QW $= z$, WH $= \sqrt{r^2 - z^2}$, KH $= 2 \times \sqrt{r^2 - z^2}$, and $\overline{KH}^3 \dot{z} = 8 \times \overline{r^2 - z^2}|^{\frac{3}{2}} \times \dot{z}$; the fluent of which quantity, while z increases from 0 to r is $\frac{8 \times 3\pi r^4}{16}$,* and for both semicircles, the

* Fluent of $\overline{r^2 - z^2}|^{\frac{3}{2}} \dot{z} =$ fluent of $r^2 \times \overline{r^2 - z^2}^{\frac{1}{2}} \dot{z} -$ fluent of $\overline{r^2 - z^2}^{\frac{1}{2}} z^2 \dot{z}$.
Fluent of $r^2 \times \overline{r^2 - z^2}^{\frac{1}{2}} \times \dot{z} = r^2 \times$ the area QBHW. (fig. 23.)

fluent of $\overline{KH}^3 \times \dot{z} = 3\pi r^4$; and because $PQ = ln$, and the area of the circle AIBHSA is πr^2, the volume of the part immersed V is $= \pi r^2 ln$; moreover $GP = \frac{l}{2}$; and $OP = \frac{ln}{2}$; wherefore $GO = \frac{l - ln}{2} = d$: and since the $\frac{\text{fluent of } \overline{KH}^3 \times \dot{z}}{12V} = \frac{3\pi r^4}{12\pi r^2 ln}$; making $\frac{\text{fluent of } \overline{KH}^3 \times \dot{z}}{12V} = d$, in order to obtain the limit or limits which separate the cases of permanent and unstable equilibrium, we obtain the equation $\frac{3\pi r^4}{12\pi r^2 ln} = \overline{\frac{l - ln}{2}} \times$ or $\frac{r^2}{2l^2} = n - n^2$; $n^2 - n = -\frac{r^2}{2l^2}$; or if $2r$ is put $= b =$ the diameter of the base, $n^2 - n = \frac{-b^2}{8l^2}$ and $n = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{b^2}{8l^2}}$.

If therefore the diameter of the base bears to the axis a greater proportion than that of $\sqrt{2}$ to 1, no value can be given to the solid's specific gravity, which will cause it to float in a state of insensible equilibrium; or in other words, there is no specific gravity separating the cases in which the cylinder will float permanently, from those in which it will overset when the

$-$ Fluent of $\overline{r^2 - z^2}^{\frac{3}{2}} z^2 \dot{z} = \frac{r^2 z^2}{8} \times \sqrt{\frac{r^2 - z^2}{z^2}} - \frac{2z^4}{8} \times \sqrt{\frac{r^2 - z^2}{z^2}} + \frac{r^4}{8} \times \frac{\text{arc HS}}{r}$.

This quantity ought to be $= 0$, when $z = 0$; wherefore the entire fluent of $\overline{r^2 - z^2}|^{\frac{3}{2}} \dot{z} = r^2 \times \text{area QBWH} + \frac{r^2 z^2}{8} \times \sqrt{\frac{r^2 z^2}{z^2}} - \frac{2z^4}{8} \times \sqrt{\frac{r^2 - z^2}{z^2}} + \frac{r^4}{8} \times \frac{\text{arc HS}}{r} - \frac{\pi r^4}{16}$, because the arc $\frac{HS}{r} = \frac{\pi}{2}$ when $z = 0$, or SH = SB; when $z = r$, this fluent, that is, the fluent of $\overline{r^2 - z^2}|^{\frac{3}{2}} \dot{z}$ while z increases from 0 to r is $= r^2 \times \text{area SBQ} - \frac{\pi r^4}{16} = \frac{\pi r^4}{4} - \frac{\pi r^4}{16} = \frac{3\pi r^4}{16}$.

axis is placed vertically; the cylinder, under these circumstances, must always float permanently with its axis vertical.

When the diameter of the base bears to the length a less proportion than that of $\sqrt{2}$ to 1, two values of the specific gravity may always be assigned, which will be the limits of the cases in which the solid floats with stability or oversets; *i. e.* $n = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{b^2}{8l^2}}$. If the specific gravity should be given, the proportion of the cylinder's length to the diameter of the base may be defined which limits the cases of stability or instability of floating with the axis vertical; for since $n - n^2 = \frac{b^2}{8l^2}$, it follows that $\frac{b}{l} = \sqrt{8n - 8n^2}$; consequently n being given, if the diameter of the base should be to the length of the axis in a greater proportion than that of $\sqrt{8n - 8n^2}$ to 1, the solid will float permanently with the axis upward; but if the base should be to the length of the axis in a less proportion than that of $\sqrt{8n - 8n^2}$ to 1, the solid will overset. Thus if n* $= \frac{3}{4}$, $\sqrt{8n - 8n^2} = \sqrt{\frac{3}{2}} = 1.2247$; if therefore the diameter of the base should be in a greater proportion to the length of the axis than 1.2247 to 1, it will float permanently with the axis vertical, if in a less proportion, it will overset from that position.

Suppose a parabolic conoid CEDK (fig. 24.) of given dimensions and specific gravity, should be placed on the surface of a fluid with the vertex downward, and the axis vertical; to ascertain the limits (depending on the length of the axis, the parameter of the parabola from which the conoid is formed, and the specific gravity,) which separate the cases in which the solid

* Page 3.

will float permanently with the axis vertical, or will overset, the plane of the base being supposed perpendicular to the axis: Let CED represent a plane section of this solid passing through the axis, which section will therefore be a parabola. Suppose the specific gravity to be such as causes the solid to sink to the depth FE. AIBHA represents a circular section of the solid which coincides with the fluid's surface; draw any diameter HI, and the diameter AB perpendicular to HI. Through any point W, in the radius FH, draw the ordinate KM perpendicular to FH; and suppose the solid to be moveable round an axis of motion parallel to the diameter HI; put the parameter of the parabola $= p$; the length of the axis KE $= a$, FW $= z$; the specific gravity $= n$; $\pi = 3.14159$; also let G be the centre of gravity of the solid, and O the centre of gravity of the part immersed. Then, since the volume immersed AEB is to the volume CED as $\overline{AB}^2 \times EF$ is to $\overline{CD}^2 \times EK$, or as $\overline{EF}^2$ to $\overline{EK}^2$; and since the volume immersed AEB is to the volume CED as n to 1, it follows that as $\overline{EF}^2 : \overline{EK}^2 = a^2 : : n$ to 1, and therefore $EF = a\sqrt{n}$, and $\overline{FB}^2 = pa\sqrt{n}$; referring to the expression for determining the stability of floating bodies when the inclinations from a position of equilibrium are very small, or $\frac{\text{fluent of } \overline{KM}^3 \dot{z} \times s}{12V} - ds$, we have, applicable to the present case, the entire fluent of $\overline{KM}^3 \dot{z} = 3\pi \times \overline{FB}^4$; or, because $\overline{FB}^4 = p^2 a^2 n$, the fluent of $\overline{KM}^3 \dot{z} = 3\pi p^2 a^2 n$: V or the volume immersed $= \frac{\pi a^2 pn}{2}$; and since by the properties of the figure, $GE = \frac{2a}{3}$ and $OE = \frac{2a\sqrt{n}}{3}$, we have $GO = \frac{2a - 2a\sqrt{n}}{3} = d$, these substitutions being made in the general value, $\frac{\text{fluent of } \overline{KM}^3 \dot{z} \times s}{12V} - ds$; this quantity becomes $= \frac{3\pi p^2 a^2 n \times 2 \times s}{12 \times \pi a^2 pn}$

$-\frac{\overline{2a - 2a\sqrt{n} \times s}}{3}$, which being put $= 0$, in order to obtain the limiting value required, we obtain $\frac{p}{2} - \frac{2a - 2a\sqrt{n}}{3} = \frac{3p - 4a - 4a\sqrt{n}}{6} = 0$, and $\sqrt{n} = \frac{4a - 3p}{4a}$; consequently $\sqrt{n} : 1 :: a - \frac{3p}{4} : a$.

From this determination it appears, that if the axis should be to the parameter in a proportion less than that of 3 to 4, no specific gravity can be given to the solid which will make it float in the equilibrium, which is the limit between the stability and instability of floating; secondly, if the specific gravity of the solid bears a greater proportion to that of the fluid than the proportion which the square of the difference between the axis and $\frac{3}{4}$ of the parameter bears to the square of the axis; when the axis is placed vertical, the solid will float with stability in that position; and thirdly, if the specific gravity of the solid bears a less proportion to the specific gravity of the fluid than that which the square of the aforesaid difference bears to the square of the axis, the solid will overset when placed on the fluid with the axis vertical, and will settle permanently with the axis inclined to the vertical line. These limits agree precisely with those which are demonstrated by ARCHIMEDES, in the second book of his tract, intituled *De iis quæ in humido vehuntur,** prop. iii. and prop. iv.

* The demonstrations of ARCHIMEDES, which relate to the parabolic conoid, are founded on a supposition that this solid is generated by the revolution of a rectangular parabola on its axis; that is, of a parabola which is the section of a rectangular cone; in which case the line, called by the author (or rather by his translator, the original of this treatise being lost) "ea quæ usque ad axem," is half the principal parameter, being equal to the perpendicular distance between the plane which touches the cone, and the plane parallel to it, which is coincident with the parabola. This solid is termed by ARCHIMEDES, "conois rectangula," but the limitation appears to be unnecessary, because the demonstrations of the author are equally applicable to a solid generated by

If the specific gravity of the parabolic conoid should be less than the limit which has just been investigated, and if the axis should be to the parameter in a proportion greater than that of 3 to 4, and less than that of 15 to 8, it will float permanently on the fluid with the axis inclined to the horizon, and with the base wholly extant above the surface at some angle less than 90°; which angle may be determined by the following geometrical construction, subject to the limitation which will appear from the construction itself, or rather from the computation founded upon it.

Let ASBTD (Tab. VI. fig. 25.) represent a section of the parabolic conoid which passes through the axis; which section will be a parabola. Let the axis BE be divided into three equal parts, one of which is EF. By the properties of this figure, F will be the centre of gravity of the solid. In the line FB take FH equal to half of the parameter, and through H draw the indefinite line ΓGZ perpendicular to BE, and in the line GZ take HK = FB; in the line HΓ take HI, which shall be to HK in the proportion of the specific gravity of the solid to that of the fluid; and bisect IK in the point L; with the centre L and radius LI describe the semicircle KOI, intersecting the axis BE in the point O; through O draw OC parallel to KI, intersecting the parabola in the point C, and let PCN be drawn touching the parabola in the point C. Through C draw the indefinite line CR parallel to BE, intersecting the line KI in the point G; in

the revolution of a parabola, which is the section of any cone, whatever may be the angle at the vertex, half the parameter being substituted instead of the line, called by Archimedes "ea quæ usque ad axem;" and it is a property of conics easily demonstrable, that any parabola being given, a similar and equal parabola may be formed from the section of any cone, whatever may be the angle at the vertex, the axis being of sufficient length.

the line CR take GQ equal to half GC; and through Q draw SQT parallel to PCN. When the conoid floats permanently and at rest, the surface of the fluid will coincide with the line SQT, and the axis will be inclined to the horizon at the angle ONC: through the points F and G draw the indefinite line FGM.

The order of the demonstration will be as follows. First, to shew that, according to the construction, the volume of the immersed part SCBT is to the whole magnitude of the solid in the proportion which the specific gravity of the solid bears to that of the fluid: secondly, to shew that the centre of gravity of the solid and the centre of gravity of the part immersed are in the same vertical line; and consequently the construction will place the solid in a position of equilibrium: thirdly, to demonstrate that the equilibrium so constituted is that of stability.

Since by the properties of the circle, HI is to HK as the square of HO is to the square of HK; and the square of HO is to the square of HK as the square of CQ ($= \frac{3}{2} \times$ HO) is to the square of BE ($= \frac{3}{2}$ BF): therefore, since by the construction the specific gravity of the solid is to that of the fluid as HI to HK, it follows, that as the specific gravity of the solid is to the specific gravity of the fluid, so is the square of CQ to the square of BE: but by the properties of the parabolic conoid the magnitude of the segment SCBT is to the magnitude of the whole solid ACBTD as the square of CQ to the square of BE; and consequently it is proved that when the solid floats according to the position described in the construction, the volume immersed SCPT will be to the whole magnitude as the specific gravity of the solid

is to that of the fluid, which was in the first place to be demonstrated. Secondly, because CQ is the abscissa of the segment SCT corresponding to the vertex C and ordinate SQ, and by the construction CG = 2 GQ, it follows from the properties of the solid that G is the centre of gravity of the segment or part immersed SCBT. By the properties of the parabola, as ON is to CO so is CO to half the parameter, that is, as ON : CO :: CO = GH : FH; therefore since the triangles GHF, CON, have one right angle each, and the sides round the equal angles are proportional, the triangles will be similar; consequently the angle OCN = the angle NFG: the sum of the angles FNC, NFC, is therefore a right angle, and the line FGM is perpendicular to the horizontal line PCN; and since F by construction is the centre of gravity of the parabolic conoid, and G has been proved to be the centre of gravity of the part immersed, and the line FGM is vertical, it follows, that the centres of gravity of the entire solid and of the part immersed are in the same vertical line, and consequently the solid is in a position of equilibrium, according to the construction. Thirdly, this equilibrium is that of stability; for let the solid be conceived to be turned round an axis passing through the centre of gravity, through a small angle, in such a direction as to depress the parts towards D, and to elevate those near to A; in that case the lowest point of the curve will be situated between C and B; suppose it to be at W, draw WX = CQ, parallel to BE, and take Wg = $\frac{2}{3}$ of WX. Then since* $\overline{CQ}^2$ is to $\overline{BE}^2$ as the specific gravity of the solid to that of the fluid, it is evident that however the axis BE is inclined to the horizon, $\overline{CQ}^2$ and consequently CQ must

* Page 53.

always continue of the same value, and therefore $\frac{2}{3}$ of CQ $= \frac{2}{3}$ of WX or CG $=$ Wg; consequently g is the centre of gravity of the part immersed after the inclination. And since the abscissa or portion of the diameter intercepted between the lowest point and surface of the fluid must always be of the same magnitude while the specific gravity remains the same; and by the construction Wx is made equal to the abscissa CQ; it follows, that when the solid has been so inclined, that the lowest point shall coincide with W, CG $= wg$, and consequently wg is always less than wV; if therefore a line gz is drawn through the centre of gravity g perpendicular to the horizon, the point of intersection z with the horizontal line RU will be between the points F and U; and the pressure of the fluid acting in the direction of the line gz will cause an angular motion in the solid,* which elevates the point D and depresses the point A, or, in other words, will counteract the inclination of the solid, by which it is deflected from its position of equilibrium. By the same method of argument it is shewn, that if the solid is inclined on the contrary direction, a force is created by the position of the centre of gravity of the part immersed, which restores the solid to its former situation, as found by the construction; which therefore places the solid in a position of equilibrium which is permanent.

The several conditions by which this construction is limited will be more easily deduced from analytical investigation, than from having recourse to geometrical constructions.

To represent in general terms the angle CNO, at which the axis of the solid is inclined to the horizon, let BE $= a$; 2HF or the parameter $= p$; also let the specific gravity of the solid be to that of the fluid as n to 1; consequently

* Page 18.

$FH = \frac{p}{2}$; $FB = \frac{2a}{3}$; $BH = \frac{2a}{3} - \frac{p}{2} = \frac{4a - 3p}{6}$; and since by the construction KH : HI : 1 : n, and $KH = BF = \frac{2a}{3}$, it follows that $HI = \frac{2an}{3}$, and $HO = \frac{2a\sqrt{n}}{3}$; consequently $OB = HB - HO = \frac{4a - 3p}{6} - \frac{2a\sqrt{n}}{3} = \frac{4a - 3p - 4a\sqrt{n}}{6}$; $CO = \sqrt{\frac{4a - 3p - 4a\sqrt{n}}{6}} \times \sqrt{p}$: and because $ON = \frac{4a - 3p - 4a\sqrt{n}}{3}$, it appears that CO will be to ON, that is, the tangent of the angle of inclination CNO, will be to radius as $\sqrt{\frac{4a - 3p - 4a\sqrt{n}}{6}} \times \sqrt{p}$ to $\frac{4a - 3p - 4a\sqrt{n}}{3}$, or as $\sqrt{\frac{\frac{3}{2} \times p}{4a - 3p - 4a\sqrt{n}}}$ to 1. When therefore the angle CNO becomes equal to 90°, that is, when the solid floats with the axis in a vertical position, the tangent of inclination $= \sqrt{\frac{\frac{3}{2}p}{4a - 3p - 4a\sqrt{n}}}$ becomes infinite, or, which is the same thing, $4a - 3p - 4a\sqrt{n} = 0$, and consequently $\sqrt{n} = \frac{4a - 3p}{4a}$, precisely coinciding with the limit deduced by a different method* of investigation.

But another inquiry is here suggested. It is evident that this construction is applicable only while the solid floats in such a manner that the whole of the base AD shall be extant above the fluid's surface. To know in what cases this condition takes place, it will be necessary to investigate what must be the value of the solid's specific gravity, and the proportion of the axis to the parameter when the solid floats permanently, so that the surface of the fluid shall pass through one of the extremities of the base A. The result will shew the limit, or limits, if there are more than one, which sepa-

* Page 51.

rate the cases in which the solid floats permanently with the base entirely extant above the fluid's surface, from those in which a part of the base is immersed under it.

The notation remaining as before, since OB or BN $= \frac{4a - 3p - 4a\sqrt{n}}{6}$ and EB $= a$, (fig. 26.) by addition EN $= \frac{10a - 3p - 4a\sqrt{n}}{6}$; and because NW $=$ CQ* $= a\sqrt{n}$, it follows that EW $=$ EN $-$ NW $= \frac{10a - 3p - 10a\sqrt{n}}{6}$: and since AE $= \sqrt{ap}$, the tangent of the angle CNO or AWE, is to radius, as EA to EW, or as $\sqrt{ap} : \frac{10a - 3p - 10a\sqrt{n}}{6}$, that is, making the radius $= 1$, the tangent of the angle AWE or CNO $= \frac{\sqrt{ap} \times 6}{10a - 3p - 10a\sqrt{n}}$; but the tangent† of CNO $= \sqrt{\frac{\frac{3}{2} \times p}{4a - 3p - 4a\sqrt{n}}}$, which two quantities are therefore equal, or $\frac{\sqrt{ap} \times 6}{10a - 3p - 10a\sqrt{n}}$ $= \sqrt{\frac{\frac{3}{2}p}{4a - 3p - 4a\sqrt{n}}}$; or if $1 - \sqrt{n}$ is put $= m$, $\frac{\sqrt{ap} \times 6}{10ma - 3p}$ $= \sqrt{\frac{\frac{3}{2} \times p}{4ma - 3p}}$; and by squaring both sides, $\frac{36ap}{100m^2a^2 - 60map + 9p^2}$ $= \frac{3p}{2 \times \overline{4ma - 3p}}$, or $\frac{24a}{100m^2a^2 - 60map + 9p^2} = \frac{1}{4ma - 3p}$, which is reduced to the equation, $100m^2a^2 - 60m.ap + 9p^2 = 96ma^2 - 72ap$; or $m^2 - \frac{60pa + 96a^2}{100a^2} \times m = \frac{-9p^2 - 72ap}{100a^2}$. Wherefore m $= \frac{30pa + 48a^2}{100a^2} \pm \sqrt{\overline{\frac{30ap + 48a^2}{100a^2}}^2 - \frac{9p^2 + 72ap}{100a^2}} = \frac{30p + 48a}{100a}$

* By the preceding investigation in appears, that HO $= \frac{2a\sqrt{n}}{3}$, and since HO $=$ GC and GQ $= \frac{1}{2}$ GC, it follows that CQ or NW $= a\sqrt{n}$.

† Page 56.

$\pm\sqrt{\frac{576a^2 - 1080ap}{2500a^2}} = \frac{15p + 24a \pm \sqrt{576a^2 - 1080pa}}{50a}$; consequently, restoring the value of $m = 1 - \sqrt{n}$, $1 - \sqrt{n} = \frac{15p + 24a \pm \sqrt{576a^2 - 1080pa}}{50a}$; and therefore $\sqrt{n} = \frac{26a - 15p \pm 6\sqrt{2a} \times \sqrt{8a - 15p}}{50a}$.

Various inferences follow from this determination. In the first place, although the object of the preceding investigation was, to find a single value only of the specific gravity, which would cause the solid to float permanently with the extremity of the base coincident with the fluid's surface, yet by the result it appears, that there are two values of the specific gravity which will answer this condition under a certain limitation, which is also discovered by the solution; this is, that the axis (a) shall be to the parameter (p) in a proportion greater than that of 15 to 8; for if that proportion should be less, $8a$ will be less than $15p$; in which case the quantity $\sqrt{8a - 15p}$ becomes impossible. From which circumstance it may be inferred, that whenever the axis is to the parameter in a less proportion than of 15 to 8, the solid will float permanently on the fluid with the whole of the base extant above the fluid's surface, whatever may be the specific gravity of the solid. This limit is precisely the same with that which is demonstrated by Archimedes, in the second book of his tract, intituled *de iis quæ in humido vehuntur*, prop. vi. When the axis bears a greater proportion to the parameter than that of 15 : 8, the solid will float either with the base entirely out of the fluid, or partly immersed under it, according to the specific gravity. Having given the axis a in a greater proportion to the parameter p than 15 to 8, by making the

specific gravity $n = \left(\frac{26a - 15p + 6 \times \sqrt{2a} \times \sqrt{8a - 15p}}{50a}\right)^2$ or $n = \left(\frac{26 - 15 - 6 \times \sqrt{2a} \times \sqrt{8a - 15p}}{50a}\right)^2$, the specific gravity of the fluid being $= 1$, the solid will float with the extremity of the base in contact with the fluid's surface. If the specific gravity is greater than $\left(\frac{26a - 15p + 6 \sqrt{2a} \times \sqrt{8a - 15p}}{50a}\right)^2$, or less than $\left(\frac{26a - 15p - 6 \times \sqrt{2a} \times \sqrt{8a - 15p}}{50a}\right)^2$, the solid will float with the base wholly above the surface. If the specific gravity of the solid is to that of the fluid in any proportion between the limits $\left(\frac{26a - 15p + 6 \times \sqrt{2a} \times \sqrt{8a - 15p}}{50}\right)^2$ to a^2, and $\left(\frac{26a - 15p - 6 \times \sqrt{2a} \times \sqrt{8a - 15p}}{50}\right)^2$ to a^2, the solid will float with the base partly immersed beneath the fluid's surface.

These limits are determined by geometrical construction in the treatise before quoted (lib. 11. prop. x. *et seq.*) to which construction the preceding investigation may serve as a comment and analysis; and some elucidation of this kind may perhaps be deemed the more requisite, since no traces are to be found in the work referred to of the method of investigation or train of reasoning, by which a problem of so much difficulty was solved, without assistance from analytical operations, at least from any that would seem competent to such an inquiry.*

* Before any proposition can be demonstrated synthetically, it must have been investigated or discovered by some previous train of reasoning: it has been supposed that the ancient geometricians purposely concealed the analysis of their propositions; but as no satisfactory evidence is produced to support this conjecture, it is probable that the supposed concealment arose from the want of a proper notation, by which analytical investigations might be conveniently expressed.

This construction of Archimedes* may be justly regarded as one of the most curious remains of the ancient geometrical synthesis, and is here inserted, in order that the agreement between the solutions by analytical investigation and geometrical construction, may appear in the most satisfactory point of view.

Having given the parabola APBL, (fig. 27.) which is a section of a conoid passing through the axis BD, and having given the axis BD, which is to the parameter in a greater proportion than 15 to 8, it is required to express, by geometrical construction, the two proportions which the specific gravity of the conoid must bear to that of the fluid, so that the solid may float permanently on the fluid when the surface passes through an extremity of the base.

BD represents the axis of the conoid, DA is the greatest ordinate to the axis; join the points B and A, and bisect BA in T; draw TH perpendicular to AD; and with the axis TH, and ordinate AH, describe the parabola ATD; in the axis BD set off DK $= \frac{1}{3}$ of DB, and make KR $= \frac{1}{2}$ the parameter; also set off KC to DB in the proportion of 4 to 15: consequently DB bears a greater proportion to KR than 15 to 4; and since KR is half the parameter, it follows that the axis is to the parameter in a greater proportion than that of 15 to 8. Through C draw CE parallel to DA intersecting BA in E, and draw EZ perpendicular to AD. With the ordinate AZ and axis ZE describe the parabola AEI, and through R draw the line RGY, intersecting the parabola AEI in the points G and Y; through the points G and Y draw the lines ON, PQ, perpendicular to AD, intersecting the parabola ATD in the points X and F.

* *De iis quæ in humido vehuntur,* Lib. ii. prop. x.

Then the proposition affirms that the solid will float permanently on the fluid with the surface thereof in contact with one extremity of the base, when the specific gravity of the solid is to that of the fluid as the square of the line OX is to the square of the axis BD, or as the square of the line PF is to the square of the axis BD.

Instead of inserting the geometrical demonstration of this construction, it will be more expedient, in the present instance, to proceed by a contrary method of argument, *i. e.* by assuming the construction as true, and inferring from it the proportions of the specific gravities in question, and comparing the proportions so inferred with those which have been already found by analytical investigation. Proceeding, according to this method, through the points X and O draw the lines SX, OY, parallel to AD; and since the axis $DB = a$, and $DK = \frac{a}{3}$ by construction, and $KC = \frac{4a}{15}$, it follows that $DC = \frac{9a}{15}$, and $BC = \frac{6a}{15}$; moreover, by the properties of the parabola $DA = \sqrt{ap}$, and the triangles ABD, ECB, being similar, $EC = \frac{DA \times BC}{BD} = \frac{\sqrt{ap} \times 6}{15} = ZD$; moreover, as DB : ZE or DC : : DA : ZA, that is, as $a : \frac{9a}{15} : : \sqrt{ap} : ZA = \frac{9\sqrt{ap}}{15}$; and consequently the parameter of the parabola $AEI = \frac{\overline{AZ}^2}{ZE} = \frac{\overline{AZ}^2}{DC} = \frac{9 \times 9 \times ap}{15 \times 15} \times \frac{15}{9a} = \frac{9p}{15}$. And because $RC = EM = KC - KR = \frac{8a - 15p}{30}$, and $\overline{ZN}^2 =$ parameter of the parabola $AEI \times ME$, it follows that $ZN = \sqrt{\frac{ME \times 9p}{15}} = \sqrt{\frac{8pa - 15p^2}{50}}$, and $ND = ZD - ZN = EC - ZN = \frac{\sqrt{ap} \times 6}{15} - \sqrt{\frac{8ap - 15p^2}{50}} = \frac{\sqrt{8ap} - \sqrt{8ap - 15p^2}}{\sqrt{50}}$,

and BY $= \frac{ND^2}{p} = \frac{16a - 15p - 4\sqrt{2a} \times \sqrt{8a - 15p}}{50}$, and YD = ON $= a - \frac{16a - 15p - 4\sqrt{2a} \times \sqrt{8a - 15p}}{50} = \frac{34a + 15p + 4\sqrt{2a} \times \sqrt{8a - 15p}}{50}$; and since HN = HD* — ND, we shall obtain HN = HD $=$ HA $= \frac{\sqrt{ap}}{2} - \frac{\sqrt{8ap} - \sqrt{8ap - 15p^2}}{\sqrt{50}} = \frac{\sqrt{ap} + \sqrt{16ap - 30p^2}}{10}$, and TS $= \frac{\overline{HN}^2}{\text{parameter of ATD}} = \frac{17a - 30p + 2\sqrt{2a} \times \sqrt{8a - 15p}}{50}$; wherefore since TH $= \frac{a}{2}$, and NX = TH — TS, it follows that NX $= \frac{a}{2} - \frac{17a - 30p + 2\sqrt{2a} \times \sqrt{8a - 15p}}{50} =$ $\frac{8a + 30p - 2\sqrt{2a} \times \sqrt{8a - 15p}}{50}$; or finally, OX = ON — NX = $\frac{34a + 15p + 4 \times \sqrt{2a} \times \sqrt{8a - 15p}}{50} - \frac{8a + 30p - 2 \times \sqrt{2a} \times \sqrt{8a - 15p}}{50}$ $= \frac{26a - 15p + 6\sqrt{2a} \times \sqrt{8a - 15p}}{50}$.

By a computation similar to the preceding it is found, that the line PF $= \frac{26a - 15p - 6\sqrt{2a} \times \sqrt{8a - 15p}}{50}$.

It is therefore a consequence, from the geometrical construction assumed as true, that the parabolic conoid will float permanently with the extremity of the base in contact with the fluid's surface; if the specific gravity of the solid is to that of the fluid, either in the proportion of

$\left|\frac{26a - 15p + 6 \times \sqrt{2a} \times \sqrt{8a - 15p}}{50}\right|^2$ to a^2, or in that of

$\left|\frac{26a - 15p - 6 \times \sqrt{2a} \times \sqrt{8a - 15p}}{50}\right|^2$ to a^2; precisely agreeing with the proportions which were deduced from analytical investigation: †

* HD = HA $= \frac{\sqrt{ap}}{2}$.

† Page 59.

by which agreement both the construction and investigation receive the most satisfactory confirmation.

It has been observed in the course of the preceding pages, that the theorems* investigated to discover the floating positions of bodies, are no less applicable to ascertain the stability of floating, or the resistance which the fluid's pressure opposes to any force applied to incline a floating body from its position of equilibrium. This latter branch of statics is a subject deserving of every attention which science and practical experience can bestow upon it, from the immediate relation it bears to the motion and equilibrium of ships at sea. By this principle, the wind's impulses become effectual in propelling vessels, which, in default of stability, are rather inclined from the perpendicular than moved forward by the force of the wind: and when a ship has been nearly overset by the violence of the elements, it is the power of stability which still sustains, and (if sufficient) at length restores it to the upright position.

The stability of a floating body when inclined through any angle from the perpendicular, has been obtained by investigating a general value of the perpendicular distance GZ† = $\frac{bA}{V} - ds$; (fig. 2.) for the distance between the two vertical lines, one of which passes through the centre of gravity of the solid, and the other through the centre of gravity of the volume immersed. This principle is now to be applied to ascertain the stability of ships: this will be effected by finding either by construction or by calculation, the length of the line GZ: and if the vessel's weight should be W, the measure of stability will be GZ × W, by which it is plainly seen, that if any force M should be applied at a distance from the centre of gravity SG, (fig. 2.) and in a direction perpendicular to SG, to balance

* Page 16. † Page 15.

or counterpoise the force of stability, there will arise the equation $M \times SG = W \times GZ$.

In the particular case, when the angles at which a floating solid is inclined from the position of equilibrium are very small, the line GZ (fig. 2.) has been found* $= \frac{\text{fluent of } \overline{AB}^3 \times \dot{z} \times s}{12V} - ds$: in which expression $\dot{z}$ is a small portion of a line drawn coincident with the fluid's surface, and parallel to the axis of motion; AB is the breadth of the solid at the water's surface, corresponding to the line z parallel to the axis; V is the total displacement or volume immersed; d is the distance GO; and s the sine of the small angle of inclination from the position of equilibrium. Respecting this expression it is observable, that since $\frac{\text{fluent of } \overline{AB}^3 \times s \times \dot{z}}{12V} = ET$, (fig. 2.) and $d = OG = EG$, it follows that $\frac{\text{fluent of } \overline{AB}^3 \dot{z}}{12V} = ES$, and $\frac{\text{fluent of } \overline{AB}^3 \dot{z}}{12V} - d = GS$; which quantity is invariably the same whatever may be the inclination of the floating body from the position of equilibrium, provided that inclination is very small; that is, the point S is immoveable in respect of the point G, while the floating body revolves through any different small angles round the axis, passing through the centre of gravity G in a direction perpendicular to the plane ADHB. Since, therefore, the measure of stability $GZ \times W$ is $\overline{\frac{\text{fluent of } \overline{AB}^3 \times \dot{z}\, s}{12V} - ds} \times W$ and $\frac{\text{fluent of } \overline{AB}^3 \times \dot{z}}{12V} - d = GS$, (fig. 2.) it follows that the measure of stability $= W \times SG \times s$, agreeing with the value which EULER has deduced by other methods for expressing the stability of vessels when the angles of inclination are evanescent.†

* Page 21.

† *Théorie complette de la Construction des Vaisseaux*, chap. viii.

If SG = o, that is, if the centre of gravity of the solid coincides with the point of equipoise S, otherwise called the metacentre,* or centre of equilibrium, the stability will be = o, or in other words, the solid will float in all positions alike, without effort to restore the upright position when inclined, or to incline itself further; it being remembered that the angles of inclination are very small. When the centre of gravity is situated beneath the metacentre, the solid must always float with stability, the measure of which is $W \times SG \times s$, in which case this force acts on the solid to turn it in a direction contrary to that in which it is inclined from the upright position; but when the centre of gravity is placed above the metacentre, (fig. 2.) the quantity $W \times SG \times s$ having passed through o, becomes a force which acts to turn the solid in the same direction in which it is inclined, and will therefore constitute the equilibrium of instability. The determination of the point S becomes, for these reasons, of consequence in estimating the stability of vessels and other bodies when the angles of inclination are very small, and is particularly of use in ascertaining whether a solid, when placed on a fluid in a given position of equilibrium, will float permanently in that position, or will overset. Because it depends on the stability or instability † of floating when the angles of inclination are of evanescent magnitude, whether the solid will continue to float in a position of equilibrium or will revolve on an axis until it settles in some other. These theorems, however, for the measure of stability being applicable only in those cases when the angles of inclination from the position of equilibrium are extremely small, when a ship or other body is inclined 10°, 15°, or 20°, the stability of floating is to be ob-

* Bouguer. Liv. i. sect. iii. chap. iv. † Page 21.

tained by having recourse to the theorem demonstrated in page 14, where it is shewn, that the stability of a vessel is truly measured by its weight, and the distance between the two vertical lines which pass through the centres of gravity of the vessel and the centre of gravity of the immersed volume; or if s be put to represent the sine of the angle of inclination from the perpendicular, V = the total displacement or volume immersed; A = the volume immersed in consequence of the inclination; b = the horizontal line bc; d = the line GO, (fig. 2.) and W = the weight of the vessel, the measure of the vessel's stability appears by this theorem to be $W \times GZ = \overline{\frac{bA}{V} - ds} \times W$. In applying this expression to any case in practice, it is supposed that the position of the centre of gravity of the ship, and the position of the centre of gravity of the immersed volume, when the ship floats in an upright position, are both known, and consequently the distance of those points, represented by the line GO = d, is a given or ascertained quantity. The total displacement is supposed to have been determined by previous measurements, which quantity is denoted by the letter V; and consequently the weight of a quantity of water, the volume of which is V, will be = W, or the vessel's weight. s, the sine of the angle of inclination from the upright position, is necessarily given from the nature of the case, and may be of any magnitude. The only quantity which remains to be determined, for ascertaining the measure of the vessel's stability, is bA. To facilitate this determination the following observations are premised. If a line be conceived to pass through the centre of gravity parallel to the horizon from the head to the stern, when the ship floats in an upright position, that line is termed the longer axis, to distinguish it from another line, also horizontal, which passes

through the centre of gravity in a direction perpendicular to the former, and is called the shorter or transverse axe. A vertical plane drawn through the longer axe when the vessel floats upright divides it into two parts perfectly similar and equal; in which particular the figures of ships may be termed regular; although in other respects they are of forms not restrained to any uniform proportions. From the equality of these two divisions of a vessel, it must necessarily happen that when it floats in a quiescent position the similar parts on the opposite sides will be equally elevated above the water's surface. A ship thus floating in a position of equilibrium may be conceived to be divided into two parts, by the horizontal plane which is coincident with the water's surface; and the section formed by this plane passing through the body of the vessel is termed the principal section of the water, and is represented in fig. 2. as coincident with the line AB: when the ship is caused to heel, by being inclined round the longer axe through any angle SGK or NXB, (fig. 2.) the plane in the ship represented by the line AB will be transferred to the position IN, and the section of the water will now pass through the vessel in the direction of a plane coincident with AP, inclined to the former plane in the angle NXP, and may be termed, merely for the sake of distinction, the secondary section of the water. These two planes intersect each other in the line denoted by the point X, or rather in the line which is projected into the point X on the plane ABDH. Since the vessel is supposed to be inclined round the longer axe, it follows, that the line of intersection denoted by X will be parallel to that axis. And since from the laws of hydrostatics the volume PXN, which has been immersed in consequence of the inclination, is equal to the volume IXW, which has been elevated

above the water's surface by the same cause are precisely equal, the position of the line represented by the point X (always parallel to the axis) will depend on the figure which is given to the sides of the vessel PN, WI. It has been seen that when the figure is a parallelopiped floating with two plane angles thereof immersed, the point X (fig. 6.) bisects the lines corresponding to AB or IN in fig. 2: when the same solid floats with one plane angle only immersed, (fig. 10.) the point X is removed nearer to those parts of the solid which are more immersed by the inclination. In a ship, the breadth of which continually alters from the head to the stern, and in no regular proportion expressible by geometrical laws, it is evident that the position of the point X, representing the line in which the water's surface intersects the vessel in its two positions, must be determined practically by methods of approximation, from which, at the same time, the other requisites for this solution will be obtained. Since to find the value of the quantity bA in the expression $W \times \overline{\frac{bA}{V} - ds}$, it is necessary that the position of the point X should previously be known: to determine this particular it will be expedient to conceive the volume (fig. 2. and 28.) NXP, which has been immersed in consequence of the inclination, and that which has been elevated above the fluid's surface, or IXW, to be divided into segments, by vertical planes passing perpendicular to the longer axis, and at a distance of a few feet from each other, for instance, 2 or 3 feet; each of these segments will be of a wedge-like form, (fig. 28.) contained between two planes, $XxPp$ and $XxNn$, inclined to each other at the given angle of inclination NXP; two vertical parallel planes NXP, nxp, which are nearly equal, and the portion $NPnp$, of the ship's side.

The distance between the planes NXP, *nxp*, is the line $Xx = Nn = Pp$; Xx produced, is the line in which the two sections of the water intersect each other, and is therefore coincident with the water's surface, and is parallel to the longer axis. The dimensions of the vessel being supposed known, the lines AB, NI, will be known in fig. 2: from these data the lines NX, PX, (fig. 2. and 28.) are to be assumed by estimation, and the angle NXP being given by the supposition, the area NXP is known from the rules of trigonometry, and the area PTNP may be inferred by the known methods of approximation.*

In like manner the area *xptn* is to be determined, and a mean of the two areas being multiplied into the thickness or

* Stirling. *De Interpolatione Serierum*, prop. xxxi. Chapman. *Traité de la Construction des Vaisseaux*, ch. i.

Methods of approximating to the areas of curves, founded on the differential serieses, are given by several authors, particularly by Stirling and Simpson. Admiral Chapman proposes a very ingenious method of approximation, depending on the properties of the parabola; either is sufficiently exact for the purposes of practical geometry, as appears by the instance inserted underneath: but of the two methods that of Mr. Stirling is the most correct. The two methods of approximation are severally applied and compared in the following example of finding the curvilinear area, which is comprehended between an arc of 30° and the radius, sine, and cosine of the said arc: to obtain this area by approximation, 5 equidistant ordinates are given; *i. e.* 1st. ordinate = radius = 8, 2d. $= \sqrt{63}$: 3d. $= \sqrt{60}$, 4th. $= \sqrt{55}$, 5th. $= \sqrt{48}$.

The approximate area is,

According to the method of Stirling, -	30.61153	According to the method of Chapman, -	30.61131
Correct area - -	30.61156		30.61156
Error of approximation —	.00003	—	.00025

The same method by which the areas of curves are found by approximation, may be applied with equal exactness to determine the solid contents of space, and the position of the centre of gravity.

distance Xx will be the solid contents of this segment, to a degree of exactness fully sufficient for the purposes of this approximation. In the same manner the solid contents of all the segments which are elevated above the surface are to be obtained by making $XI = AB - NX$, $XW = AB - PX$, and proceeding as in the former case. If the aggregate of the segments NXP representing the part immersed, in consequence of the vessel's inclination, should not be equal to the aggregate of the segments IAW, (fig. 2.) which are elevated above the surface, the position of the point X, or rather of the line which that point denotes, must be altered, and the same operations repeated till the sums of the segments on each side of the said line of inclination are precisely equal.

This having been effected, the magnitude of the volume immersed, denoted by A in the expression $W \times \overline{\frac{bA}{V} - ds}$, will be known; and the magnitude of each of the individual segments NXP*nxp* and IXW*ixw*, &c. will also be known; the quantity bA will be found in the following manner. The area PXNTP and its centre of gravity d are to be determined by methods of approximation. Through d draw dc perpendicular to the horizontal line PX; Xc* will be the

* The solution of problems by geometrical construction has been little practised since methods of calculation have been so much improved by the invention of logarithms and other facilities: the solutions of difficult cases are, however, sometimes obtained with sufficient exactness by construction, which would be more troublesome by any other method: in the present instance, after the area PTNP and its centre of gravity have been determined, the position of the centre of gravity d, of the entire area XNTP, and the length of the line Xc, may be most easily ascertained by the method of construction. If the line PN is bisected in the point C, the centre of gravity of the triangle PXN will be situated at the distance of $\frac{2}{3}$ CX from the point C: the centre of gravity of the triangle PXN being thus constructed with geometrical exactness, it follows, that the centre of gravity of the entire area PXNTP, which is the

distance of the centre of gravity d from the point X, estimated in the direction of the horizontal line PX.

The same operations being applied to the area $xptn$, will give the distance ex of the centre of gravity of the area $xptn$, from the point x, estimated in the direction of the horizontal line px; the mean of the two distances so found will be the distance of the centre of gravity of the solid segment XPNxpn, from the line Xx, estimated in the direction of the horizontal line XP or xp, to a degree of exactness entirely sufficient for this approximation.

Similar distances of the centres of gravity of all the segments (fig. 2. and 28.) PXNpxn, corresponding to the line Xx produced, having been found, also of all the segments IXWixw, if each of these segments is multiplied into the distance of its centre of gravity from the line Xx, estimated in a horizontal direction, the sum of the products so formed will be the value of the quantity bA in the expression $W \times \overline{\frac{bA}{V} - ds}$, which is the measure of the vessel's stability, when inclined from its upright position through an angle PXN of which the sine is to radius as s to 1: and the quantities* W, V, and d, having been previously determined, it is evident that from the methods which have been described, the vessel's stability when inclined to the given angle will be obtained.

It would be improper, in a disquisition not written on the practice of naval architecture, to enter into further detail on this subject. By what has preceded, it is evidently seen that the stability of vessels may be determined for any angles at which they are inclined from the position of equilibrium, as well as for those which are very small. In both cases

common centre of gravity of the areas PXN and PNTP, is capable of being determined with very great precision.

* Page 66.

it is necessary that the position of the centre of gravity of the ship, and that of the part immersed, when the ship floats upright, should be known; practical methods of mensuration are required, in both cases, to ascertain these points. When the angles of inclination are very small, to find the ship's stability, it is necessary to measure* the successive ordinates or breadths of the ship on a level with the water's surface, and when the angles of heeling are not limited, but are considered as being of any magnitude, the requisite mensurations are indeed more troublesome, but are not liable to more errors in execution than in the former case, when the angles are limited to those which are evanescent.

The theorems for measuring the stability of ships, which are founded on assuming the angles of inclination from the position of equilibrium evanescent, explain, in the most satisfactory manner, the principles on which the stability of ships, when heeled to small angles of inclination, is founded; they also ascertain when ships or other bodies float on the water permanently in a given position of equilibrium, or overset. But this can scarcely ever be an object of inquiry in respect of ships, which are always constructed so as to float upright, even before any ballast or lading has been added to them.

Mons. ROMME, in his valuable work on naval architecture, intituled *L'Art de la Marine*, published at Paris in the year 1787, informs his readers (p. 106), that the French ship of the line of 74 guns, called Le Scipion, was first fitted for sea at Rochfort in the year 1779. As soon as the ship was floated in deep water, a suspicion arose that she wanted stability; to ascertain this point the guns were run out on one side, and drawn in at the other; in consequence, the ship heeled 13

* CHAPMAN, chap. i. CLAIRBOIS *Architecture Navale*, part. ii. sect. i.

inches (probably meaning at the greatest measure on the side of the vessel): by adding the weight of the men brought to the same side, the depth of heeling increased to 24 inches. This being a degree of instability, which was deemed too great to be admitted in a ship of war, the ship was ordered into port, that some remedy might be applied to the defect which had been discovered. M. ROMME proceeds to relate, that a difference of opinion prevailed amongst the engineers respecting the cause of this imperfection in the ship, and the remedies by which it might be corrected. The chief engineer, who was sent from Paris to Rochfort to direct what measures ought to be adopted on this occasion, and for rectifying the like fault in two other ships of war, L'Hercule and Le Pluton, was of opinion, that the stability of the ship Le Scipion would be sufficiently increased by altering the quality and disposition of the ballast. The original ballast of the Scipio had been 84 tons of iron and 100 tons of stone; according to the new arrangement of the chief engineer, the ballast was composed of 198 tons of iron and 122 tons of stone. But as a ship of war does not admit of any alteration in the total displacement or immersed volume, to compensate for the additional weight of ballast, amounting to 136 tons, the quantity of water with which the ship had been supplied was diminished by the weight of 136 tons. This alteration must necessarily have the effect of lowering the centre of gravity of the vessel, and thereby of increasing its stability: but, on trial, this increase was by no means sufficient; the diminution of heeling measured on the vessel's side being only 4 inches. After this and other ineffectual attempts, the defect of stability was at length remedied by applying a bandage or sheathing of light wood to the exterior sides of the vessel, from 1 foot to 4 inches in

thickness, extending throughout the whole length of the water line, and 10 feet beneath it.

This account shews that the theory of stability, restrained to cases in which the angles of inclination, or heeling, are very small, cannot be relied on for ascertaining the requisite stability of ships in the practice of navigation. It must be supposed that the weight and dimensions of every part of this ship were exactly known to the engineers, yet we observe that the instability was not certainly ascertained, but suspected only to exist when the ship was first set afloat in deep water; and after this defect had been discovered by the experiment which has been related, the cause was sought for in vain, and the remedy at length was stumbled upon by accident, rather than adopted from any knowledge of the principles by which the application of it might have been directed.

It seems allowable to suppose, that if rules for ascertaining stability correspondent to any different angles of heeling, similar to those which are demonstrated in page 15, and exemplified in page 70 of this tract, had been applied to the case in question, they would have discovered that an error in the form* given to the sides of the vessel was the principal cause of the defective stability, and would have suggested the remedy accordingly; or rather would have prevented the necessity of having recourse to it, by previously shewing the original defects in the plan of the ship.

The force of stability by which ships, when inclined round

* Mr. ROMME observes, page 108, that the defect of stability in the Scipio was not occasioned by any want of breadth in the principal section of the vessel; for other ships of the same force, *i. e.* Le Magnifique, Le Sceptre, Le Minotaur, L'Intrepide, the breadths of which were the same, or rather less, than that of the Scipio, carried their sail perfectly well.

the longer axis from their position of equilibrium through different angles, endeavour to regain that position, is to be considered in two points of view respecting the motion of a vessel at sea; first, in relation to the resistance by which it opposes any force that may be applied to incline the ship, for instance, that of the wind; in which case the ship's stability, and the impulse of the wind, constitute a species of equilibrium as long as the wind continues of the same intensity. Secondly, the force of stability is to be considered as operating on the ship, after the force by which it has been inclined ceases, to restore the vessel to its upright position; the ship being continually impelled by the force of stability, revolves round an horizontal axis, passing through the centre of gravity with an increasing velocity, till it arrives at its upright position; and afterwards with a velocity continually retarded, till it arrives at the greatest inclination on the other side. This rolling of the ship, with alternate acceleration and retardation of the angular velocity, will evidently depend on the force by which the angular motion is generated; that is, on the force of stability, and its variation corresponding to the several angular distances of the vessel from its upright position; from this cause arises one of the principal difficulties in the practice of naval architecture; *i. e.* to give a vessel a sufficient degree of stability, and at the same time to avoid the inconveniences which proceed from an angular velocity of rolling, increasing and decreasing too rapidly. It is certain that the variation of the force of stability depends principally on the shape given to the sides of the vessel, which admit of being so constructed (all other circumstances permitting) that the force shall increase either slowly or rapidly to its limit.

From the preceding investigations we observe that some float-

ing bodies, during their inclination from 0° to 90°, pass through a position of equilibrium, in which the force of stability becomes evanescent: in other bodies, no limit of this kind takes place; a difference which depends partly on their forms, and partly on the disposition of the centres of gravity of the solids and of the immersed volumes. It may be satisfactory to consider, in a general view, the effects produced on the motion of ships by the different proportions of their stability while they are inclined round the longer axes. If a vessel* should be of a cylindrical form, floating with its axis horizontal, the vertical sections must necessarily be equal circles: supposing the centre of gravity of such a cylinder to be situated out of the axis, the vessel will float permanently with its centre of gravity, and the centre of the section passing through it, in the same vertical line: if such a vessel should be inclined from the upright by external force, it will be impelled in a contrary direction by the force of stability, which increases exactly in the proportion of the sine of the angle of inclination: it is plain, therefore, that a vessel of this description, during its inclination by heeling, cannot arrive at any limit where the force of stability is evanescent; on the contrary, it must continually increase until the inclination is augmented to 90°, where it will have become greater than at any other angle.

Let another case be assumed: suppose the form of the vessel to be a square parallelopiped, floating permanently with one of the flat surfaces upward; when this solid has been inclined round the longer axis through 45 degrees, the stability will be evanescent, and the least inclination greater than that angle

* This is evidently an hypothetical case, stated with a view of illustrating the subject.

will cause the vessel to overset: in this case, as the vessel is gradually inclined from the upright, the stability will first increase to a maximum, and afterwards decrease; differing altogether from the variation of the stability in the preceding case, when the vessel was supposed to be of a cylindrical form. Although vessels are usually so constructed that during any inclination from 0° to 90° they do not pass through a position of equilibrium; yet there seems reason to suppose that in some vessels the stability increases to a maximum, and afterwards decreases when the angle of inclination is farther augmented: whenever a vessel of this description should be inclined beyond the angle where the stability is greatest, the following consequence must necessarily ensue; if the angular velocity should be considerable, the rolling of the ship will be extended to large angles of inclination, because when the stability is more and more diminished as the angle of inclination is augmented, more time will be required for the diminished force to react against the ponderous mass of the vessel, in order to restore it to the upright. It is certain that the angle, as well as the celerity or slowness of rolling, depend on other elements, as well as on the stability, particularly on the weight and extent of the masts and sails, and the position of the ballast and lading: but in comparing the vibrations of the same vessel through different arcs, those elements are the same, while the force of stability alters continually as the angles of inclination are increased or diminished.

These alternate vibrations of a ship in rolling have been deemed analogous to the oscillations of a pendulum; and in order to reduce to some kind of measure so essential a quality of vessels, M. Bouguer and other writers propose to find a pendulum isochronal to the oscillations of a ship. This pro-

blem seems to imply both that the pendulum sought, and the vessel itself, shall vibrate in arcs that are extremely small; for otherwise the analogy altogether fails: no oscillating body can describe arcs of unequal lengths in equal times, unless it is impelled by forces which are in the direct ratio of the distances from the quiescent point; and therefore the oscillations of a vessel vibrating in different finite angles are evidently not isochronal with each other, since the force of stability varies in a proportion so different from that of the distances from quiescence; nor can they be isochronal with any pendulum, unless the arcs of vibration are of evanescent magnitude; in which case the force of stability being in the direct proportion of the angles of inclination from the upright, has the effect of producing an equality in the times of oscillation: to ascertain a pendulum vibrating in small arcs which is isochronal to the oscillations of a vessel, under these restrictions, is a problem which may be solved with sufficient exactness; but unless the limitation that has been mentioned should be specified, it is a question without the necessary conditions. Mons. BOUGUER* in his chapter intituled, *que les Oscillations sont Isochrones*, does not expressly mention this limitation, but we must allow it probable that he conceived it to be implied.

From the reasons that have been stated it seems to follow, that in order to form a satisfactory opinion of the qualities and performance of a vessel at sea as depending on the plan of its construction, the forces of stability at the several angles of inclination from o to the greatest limit ought to be ascertained, particularly the measure of the greatest stability, and the angle of heeling at which it takes place.

* Liv. i. sect. iii. chap. vii.

In these general remarks the water's resistance has not been considered, which must necessarily have some effect in retarding the oscillations of the vessel, and more in the larger arcs than in the smaller: it is however observable, that the resistance to the rolling of vessels is of a very different kind to that which is opposed to their progress through the water, in which case a volume of the fluid proportional to the vessel's bulk and velocity is entirely displaced during its motion; whereas in the rolling of ships a far less quantity of water suffers an alteration of place by the ship's oscillations, which is therefore the less retarded on this account.

Another observation occurs on this subject. The entire stability of a ship has been shewn to consist of the aggregate stabilities of the several vertical sections into which it can be divided. Let it be supposed that the ship has been inclined round the longer axis through a given angle, and that the vessel returns through the same angle of inclination by the force of its stability; if the forces arising from the several sections do not act in their due proportion on each side of the centre of gravity, in respect to the longer axis, the ship will not return to its position of equilibrium by revolving round the longer axis; but will be inclined round various successive horizontal lines between the longer and shorter axes; a circumstance that must create irregular motions and impulses, to which a vessel in all respects well constructed is not liable.

The theory of statics and mechanics was, I believe, first applied to explain the construction and management of vessels toward the latter end of the last century, in a work intituled *Théorie de la Construction des Vaisseaux, par* P. PAUL HOSTE, printed at Lyons in the year 1696. Several eminent mathematicians have since prosecuted this difficult subject,

particularly JOHN BERNOUILLI, BOUGUER, and the excellent M. EULER, whose treatise, intituled *Thèorie complette de la Construction & Manœuvre des Vaisseaux*, is a work correspondent to the title, entirely theoretical. In this elaborate performance the author has not only endeavoured to explain the complicated laws which influence the motion of ships at sea, but proceeds to investigate, on the ground of such data as the subject affords, the dimensions and position of the most essential parts of vessels which combine to give them every possible advantage in the practice of navigation.

Several inquiries are suggested by the perusal of these theoretical works; first, whether the proportions and dispositions of parts in ships resulting from theory have been found to differ from, or to agree with, those which had been previously established in the practice of naval architecture; secondly, if disagreement should have been discovered, whether any adequate and satisfactory trials have been made to ascertain the advantages which result from adhering to the constructions prescribed by practice, compared with those which are consequences of following the deductions from theory; and lastly, if any new forms of vessels, disposition of parts, or other varieties of construction, have been discovered by considering this subject in a theoretical view, and in what degree these inventions have been found advantageous when applied in practice.

Exclusive of the application of geometrical principles,* by

* Practical treatises on ship-building have been published by various authors, particularly by M. CLAIRBOIS, ROMME, and FRED. CHAPMAN. In these useful works theory is occasionally applied to explain and illustrate the principles of naval architecture: but no accounts are to be found in either of these volumes, as far as my researches extend, by which the construction of vessels, founded on theoretic investi-

which the forms of vessels and the disposition of their most essential parts are ascertained, theory may be considered as bearing to naval architecture a two-fold relation: first, as depending on the pure laws of mechanics, a subject on which the preceding cursory observations have been offered: secondly, the practice of naval architecture is guided, in most parts of the world, by a species of theory or systematic rule which individuals form to themselves from experience and observation alone: it is founded on the experimental knowledge in naval constructions, which has been transmitted from preceding times, combined with the more recent improvements, and includes whatever inventions of skill and ingenuity are applicable to the various machinery that is employed in the construction and management of vessels: by repeated observation on the forms, proportions, and equipment of ships, and by attention to their excellencies and defects when afloat at sea, faults are remedied, good qualities are improved, and rules of practice are by degrees established according to principles, well perceived and understood, without much assistance from the theories of mechanics, statics, and geometry, on which such principles are founded: for in this, as well as other instances, it is well known that skilful practice, aided by long experience, arrives at determinations which it is very difficult (sometimes impossible) for theory to infer: on the other hand it must be allowed, that pure theory, depending on the laws of motion, the subject of disquisition in

gation, have been subjected to practical examination during voyages. M. Chapman, in page 79 of his work (Paris edit.), expresses the proportions and disposition of parts in vessels by algebraic quantities, which, however, are not to be mistaken for deductions from theory; since the author has not pointed out any mode of investigation, or train of reasoning, by which those expressions can be deduced from the principles of mechanics.

the works of M. EULER and BOUGUER, is of great importance to the advancement of this science: for by such investigation, so far as the data are sufficient, the qualities of vessels are traced to their true causes, and are explained by general laws; whereas the principles derived from mere observation are scarcely ever applicable beyond the cases in which they have been experienced in practice.

Whatever may have been the means by which naval architecture receives progressive improvement, it seems to be generally allowed, that the art of constructing vessels has, at the present period, attained to a degree of perfection far surpassing any that has been known to former times, either ancient or modern; yet it is equally certain, that some principles, by which the construction of vessels is materially influenced, still remain to be developed and explained. It is frequently remarked by navigators, as well as by naval architects, that alterations apparently the most trivial, in the form of a vessel, in the distribution of the ballast, or in the position and extent of the masts and sails, will wholly change the qualities of a ship from bad to good, or the reverse. As these changes cannot be attributed to fortuitous causes, it is necessary to allow that they are consequences of principles certain and definite, though in many cases unknown, or imperfectly estimated by conjecture. The proportions and disposition of parts, which operate to produce good or bad effects on the sailing of ships, are probably in these instances so intricately combined as to make it scarcely possible from mere observation, however extended and diversified, to account satisfactorily for changes so remarkable: it must also be acknowledged, that some of the data on which the theory of naval architecture is founded, being imperfectly known, parti-

cularly the laws of the different resistances to the ship's motion,* it would be unsafe to rely entirely on deductions *a priori* for explaining this subject.

* The laws of resistances, opposed to bodies which move in fluids, and varying in a duplicate ratio of the body's velocities, are demonstrated by Sir ISAAC NEWTON, in the second book of the *Principia*, on conditions restrained to the particular case in which the motion of the resisted body is extremely slow, and the fluid perfectly compressed. On these conditions, the pressure which resists the motion of the body is exactly balanced by the pressure on the posterior part, and consequently the only force opposed to the body's motion, is the *inertia* of the fluid, which is displaced while the body moves through it: for the resistance of friction depending on the body's velocity must be, in a physical sense, evanescent, when the motion is very slow. It is evident, that the theory of resistances founded on these principles ought not to be applied to the solution of cases in which the velocity is much increased, without great care and circumspection; for by the increase of velocity, three different forces begin to have operation, of which the NEWTONIAN theory takes no account; *i. e.* the pressure on the anterior part of the body, the pressure on the posterior part, and the resistance of friction. The pressure on the anterior part will evidently be a constant or invariable quantity as long as the moving body continues at the same depth. The pressure on the posterior part will depend on the velocity of the body's motion, and when that velocity is = o, the pressure will be precisely equal, and contrary to that which acts on the anterior part. Moreover, when the body's velocity is equal to that with which the fluid rushes into empty space, the pressure on the posterior part will be = o, and of consequence all the pressures on the posterior surface, corresponding to the intermediate velocities, must be found between these limits. When the surfaces of the moving body are smooth, it has been supposed that the effects of friction are not very considerable. This opinion is however disproved, to the satisfaction of any one who consults the account of the very accurate and well devised experiments on the motion of bodies through the water, made under the direction of the committee of the Society for the Improvement of Naval Architecture, and published by their order. I have examined these experiments with a good deal of attention, particularly those which were made on oblong beams or parallelopipeds, denoted in the account of the experiments by the letters A, B, &c.; and find, that although the surfaces of the moving body were planed very smooth, the resistance of friction was equal to a weight of no less than ninety pounds, on a surface of 258 square feet, when the body moved with a velocity of 8 feet in a second. It appears also, by methods of calculation, founded on Sir ISAAC NEWTON's rule for drawing a parabolic line through any number of given points situate in the same plane, and applied to the above-named experiments, that the resistance of friction varies in no power of the velocity expressible by less than three di-

These difficulties will appear still greater, if it be considered that the causes which influence the motion of ships at sea are not separate and independent, but operate on each other, as well as immediately on the motion of the vessel: thus, if the position of the centre of gravity is altered by moving the ballast or lading nearer to the head or stern, this alteration will have the effect of changing the section of the water, and the form of the immersed part of the vessel; on which account, the resistance opposed by the water to the ship's motion must necessarily be changed; the centre of gravity of the part immersed will also be differently situated, which must combine

mensions thereof, that is, if z is put to denote the resistance of friction, and v to denote the velocity, the resistance requires an equation of the form $z = au + bu^2 + cu^3$; in which a, b, and c, are invariable quantities: the force also of pressure on the posterior surface is expressed by an equation equally complex: to these difficulties another is to be added, which is, that the resistance varies with the depth of the moving body, as appears by the experiments referred to. On these considerations it seems manifest, that investigations on the subject of naval architecture, founded on the theory of motion, which takes into account the resistances of the water, considering the velocity to be such as ships usually sail with, must involve algebraic expressions so complicated, as to make it very difficult, perhaps impossible, to infer any useful practical conclusions from this mode of considering the subject. EULER and BOUGUER, the principal authors who have attempted to apply the theory of resistances to naval architecture, suppose the resistance to be, according to the NEWTONIAN principles, in a duplicate ratio of the velocities; a law evidently different from that according to which vessels at sea are opposed by the medium in which they move: and one of these most eminent authors,* doubts whether this theory is not too imperfect to be relied on, when it is applied to ascertain the motion of ships at sea. Notwithstanding the impediments which arise from the complicated laws of resistance and friction, the general principles investigated in the works of these authors are no doubt capable of being applied to the solution of many difficulties which occur in considering the subject of naval architecture, due allowance being made for those irregular forces which cannot be included in the theoretic solutions.

* EULER. *Théorie complette de la Construction des Vaisseaux*, English edition, p. 93, 94.

with the alteration of the centre of gravity of the vessel, and the section of the water, to increase or diminish the stability of the ship; and it must be added, that the inclination of the masts and sails to the horizon, and the direction in which the wind impinges on them, will suffer alteration from the same cause.

Although theory alone may not be adequate to the solution of these difficulties, yet, when combined with experiments and obervations, it may be probably employed with great advantage in these researches. If the proportions and dimensions adopted in the construction of individual vessels are obtained by exact geometrical mensurations, and calculations founded on them, and observations are made on the performance of these vessels at sea; experiments of this kind, sufficiently diversified and extended, seem to be the proper grounds on which theory may be effectually applied in developing and reducing to system those intricate, subtle, and hitherto unperceived causes, which contribute to impart the greatest degree of excellence to vessels of every species and description. Since naval architecture is reckoned amongst the practical branches of science, every voyage may be considered as an experiment, or rather as a series of experiments, from which useful truths are to be inferred towards perfecting the art of constructing vessels: but inferences of this kind, consistently with the preceding remark, cannot well be obtained, except by acquiring a perfect knowledge of all the proportions and dimensions of each part of the ship; and secondly, by making and recording sufficiently numerous observations on the qualities of the vessel, in all the varieties of situation to which a ship is usually liable in the practice of navigation.

with the alteration of the centre of gravity of the vessel, and the motion of the water, to increase or diminish the stability of the ship: and it must be added, that the inclination of the [illegible], and the direction in which the wind [illegible] after alteration from the same cause.

Although theory alone may not be adequate to the solution of these difficulties, yet, when combined with experiments and observations, it may be probably employed with great advantage in these researches. If the proportions and dimensions adopted in the construction of individual vessels are obtained by exact geometrical mensurations, and calculations founded on them, and observations are made by the commanders of these vessels at sea; experiments of this kind, sufficiently diversified and extended, seem to be the proper grounds on which theory may be advantageously employed, and close and accurate reasoning, to ascertain those different, separate and hitherto unascertained causes, which contribute to impart the greatest degree of excellence to vessels of every species and description. Since naval architecture is reckoned amongst the practical branches of science, every voyage may be considered as an experiment, or rather as a series of experiments, from which useful truths are to be inferred towards perfecting the art of constructing vessels: but inferences of this kind, consistently with the preceding remarks, cannot well be obtained, except by acquiring a perfect knowledge of all the proportions and dimensions of each part of the ship; and secondly, by making and recording sufficiently numerous observations on the qualities of the vessel, in all the varieties of situation to which a ship is usually liable in the practice of navigation.

ERRATA.

Page 7, line 26, *dele* practically.

Page 16, line 5, *for* in general is, *read* in general being.

Page 20, line 8, *for* point X, *read* horizontal line drawn through the point X parallel to the axis of motion.

Page 32, lines 1 and 9, *for* WGS, *read* UGS; and line 24, *for* WGO, *read* UGO.

Page 33, line 2, *for* VW, *read* VU.

Page 40, line 26, *for* B, *read* R.

Page 46, last line, *for* QA, *read* NF.

Page 51, last line, *for* prop. iii. *read* prop. ii.

Page 52, line 18, *for* GZ, *read* ΓZ.

Page 55, line 12, *for* horizontal line, *read* indefinite horizontal line.

Page 62, line 3 and 4, *dele* HD = HA.

Page 70, line 5, *for* AB — PX, *read* WP — PX, fig. 11. and 28.

Page 79, line 11, *for* is, *read* are.

Page 82, line 8, *for* whatever may have been, *read* whatever may be.

Note to be added to page 59, last line, to the word "inquiry."

The following remark on the propositions and demonstrations of APOLLONIUS PERGÆUS, equally, or rather more applicable to those of ARCHIMEDES, is extracted from Dr. WALLIS's Algebra.

"Et quidem meritò censeri posset ille, magnus geometra, et prodigiosæ, tum phan-"tasiæ tum memoriæ vir, si possibile putemus ut potuerit ille propositiones et demon-"strationes perplexas, eo ordine quo ad nos perveniunt invenire, absque cujusmodi "aliquâ *inveniendi arte* qualis est quam nos algebram dicimus."

Dr. WALLIS's Algebra, cap. LXXVI.

Page 79, line 26, note to the words "first applied."

PERE PARDIES and Chevalier RENAUD published some partial observations on the theory of naval architecture rather before this period: but the treatise of M. L'HOSTE seems to be the first work in which this subject is considered systematically, and at length.

Fig. 1.

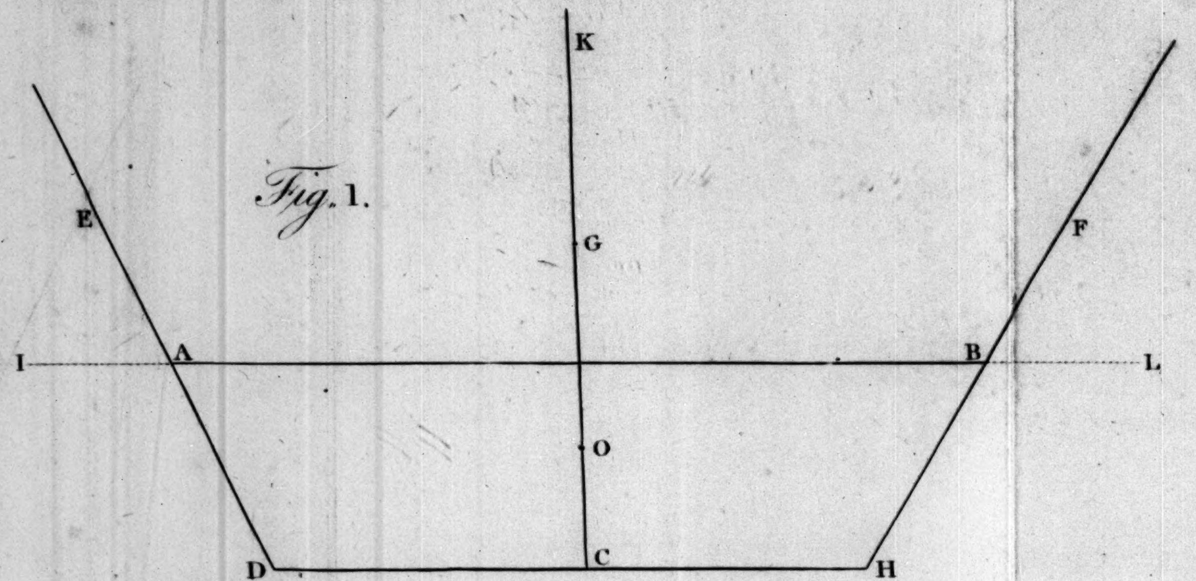
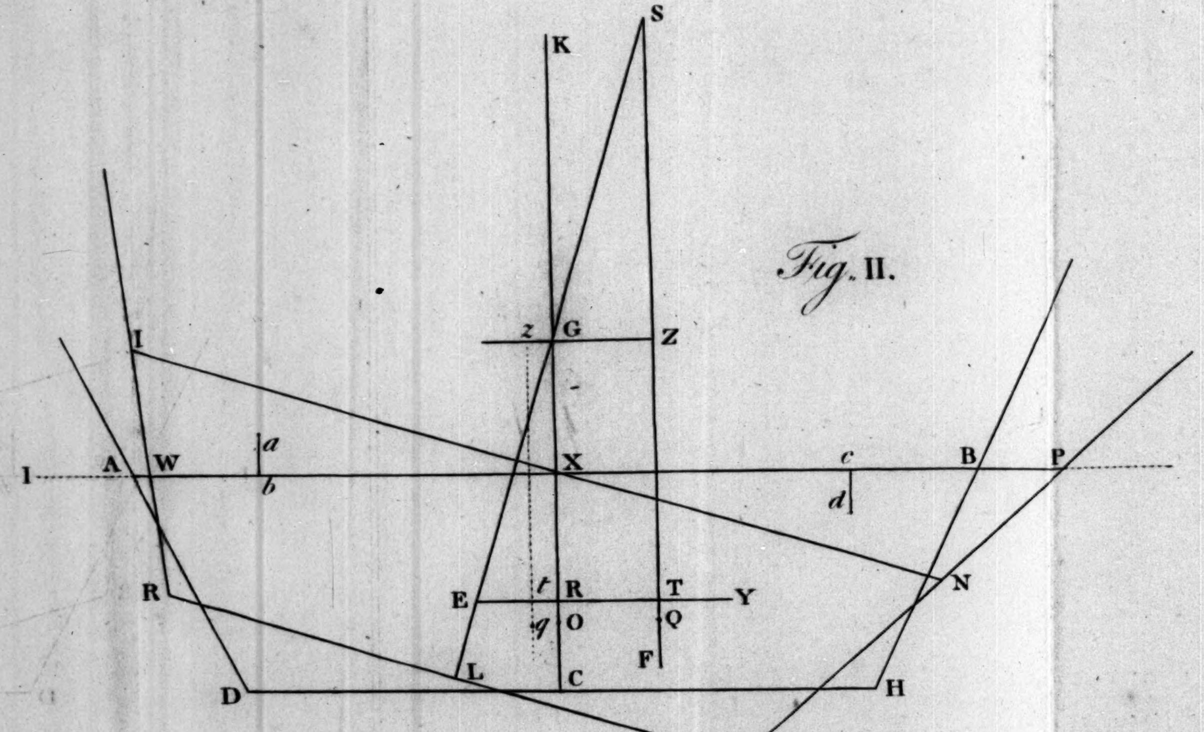


Fig. II.



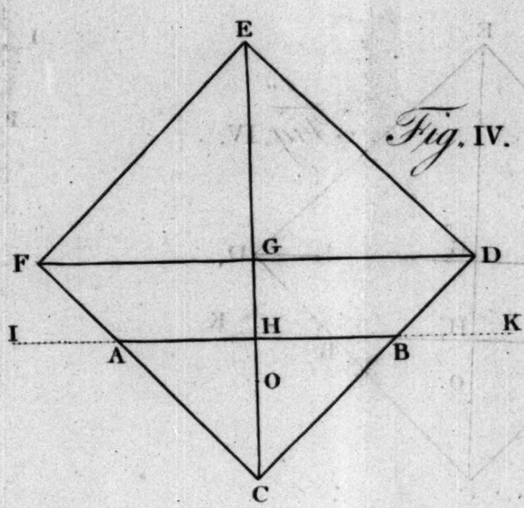
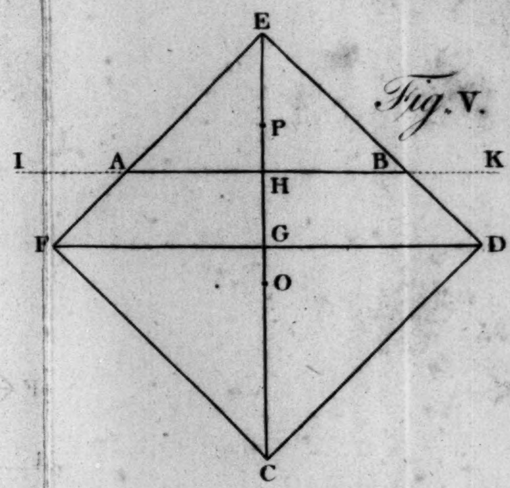


Fig. IV.



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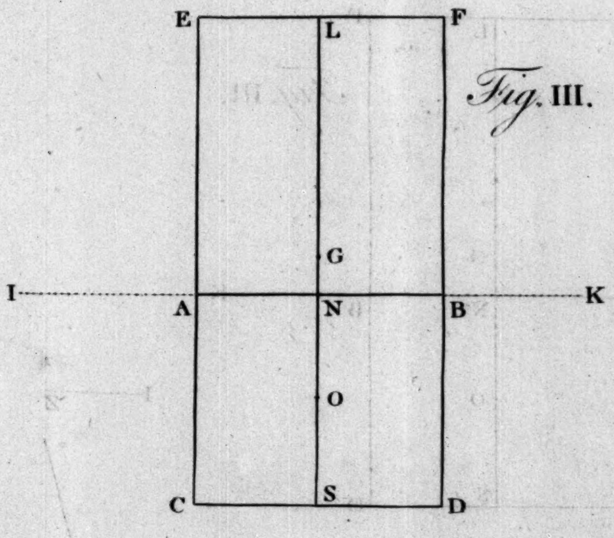

Fig. III.

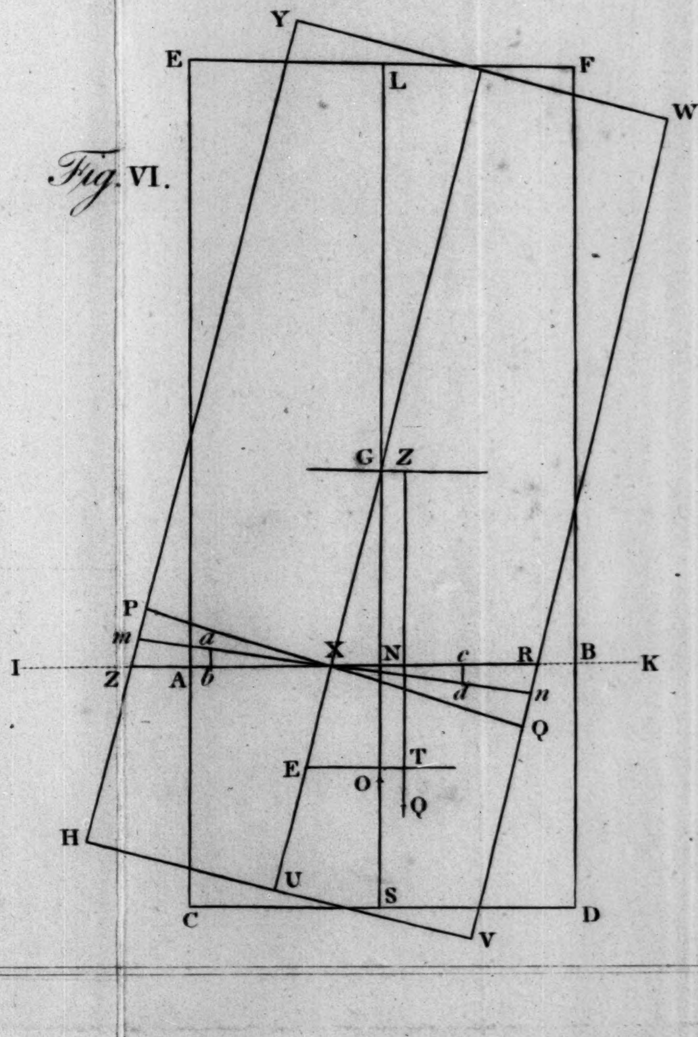

Fig. VI.

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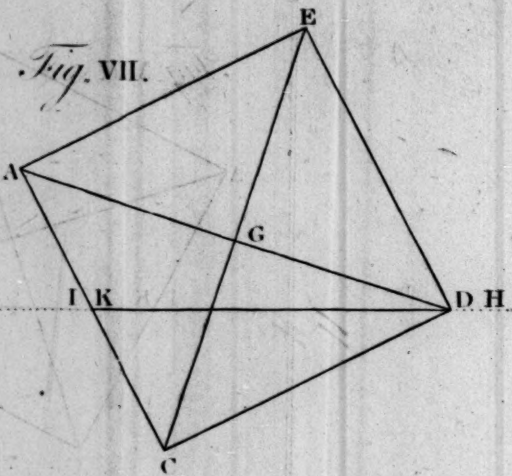
Fig. VII.



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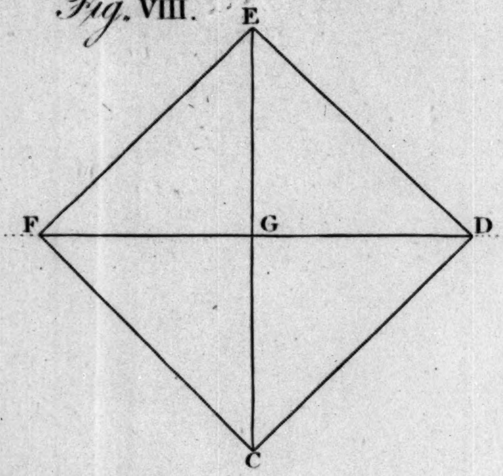

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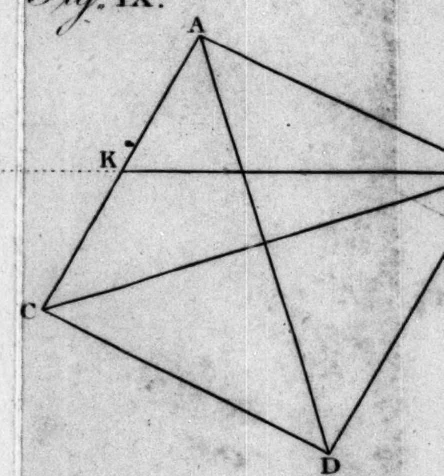

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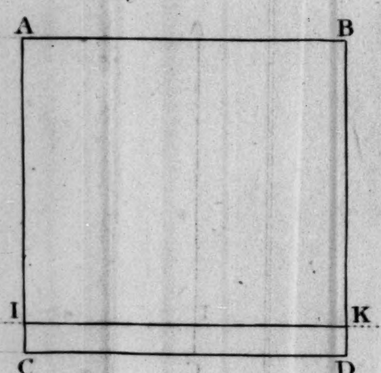

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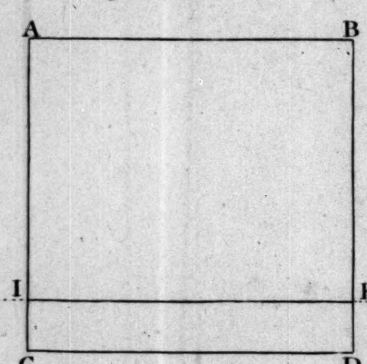

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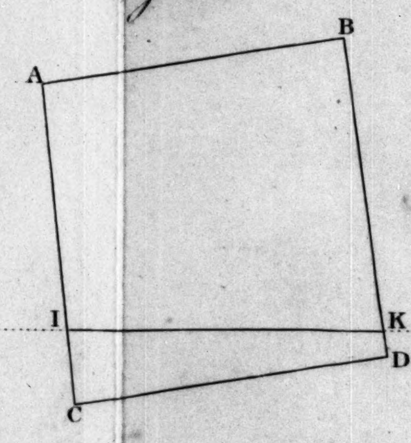

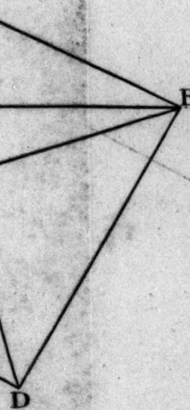

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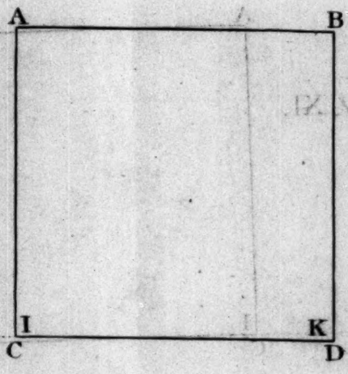
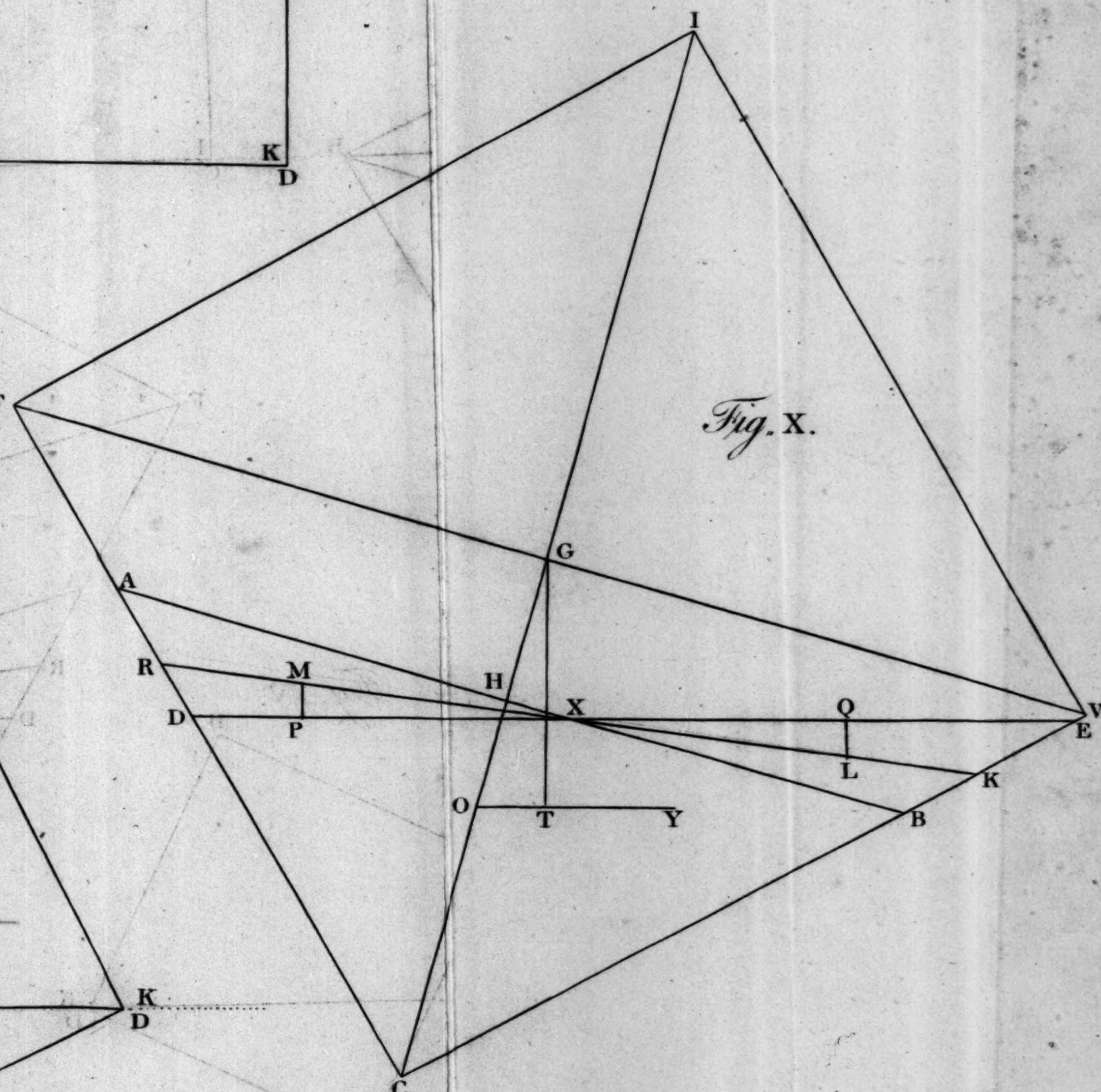


Fig. X.



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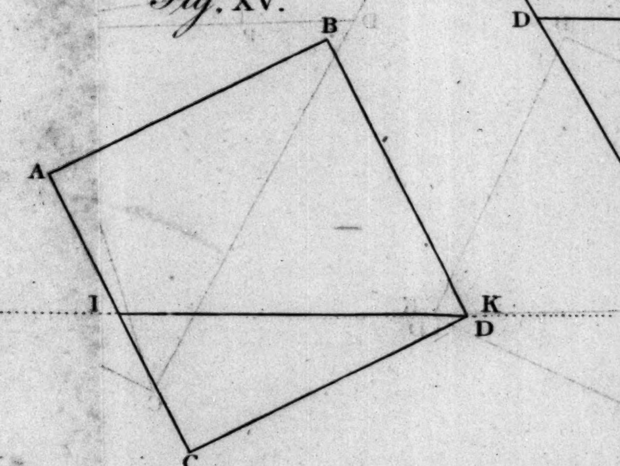

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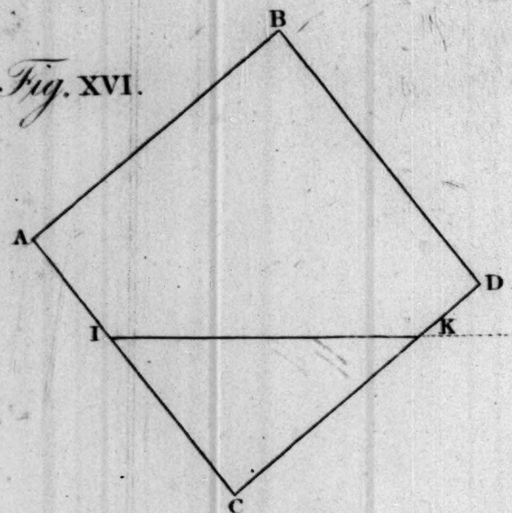
Fig. XVI.



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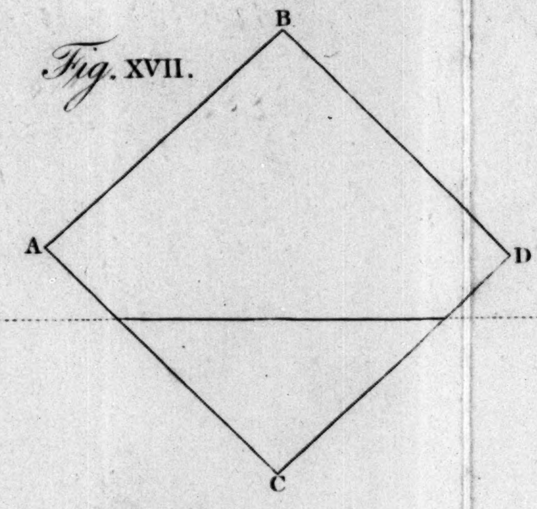

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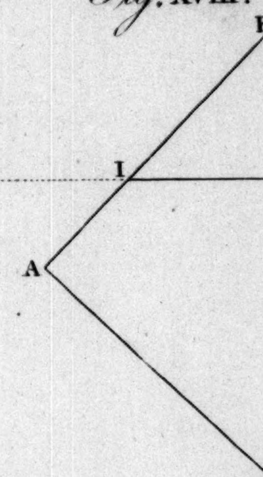


Fig. XXI.

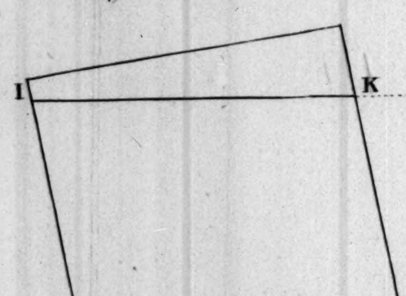
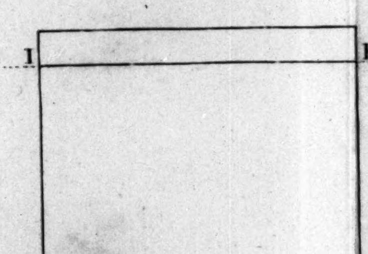
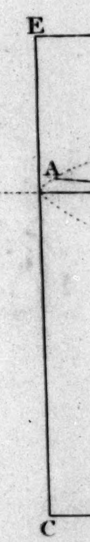


Fig. XXII.





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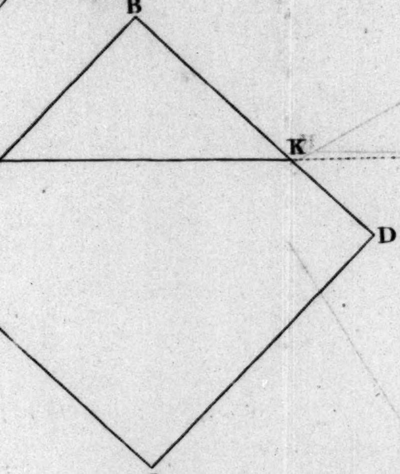

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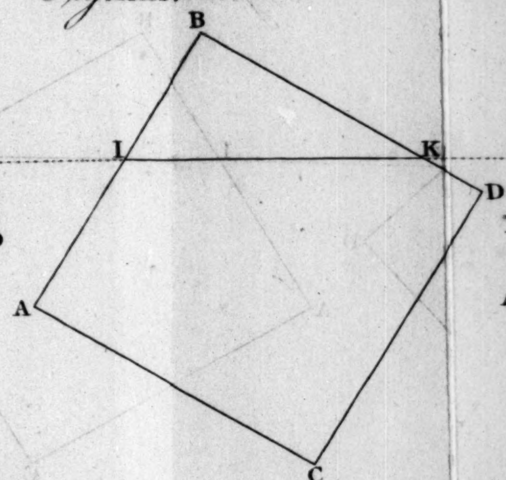

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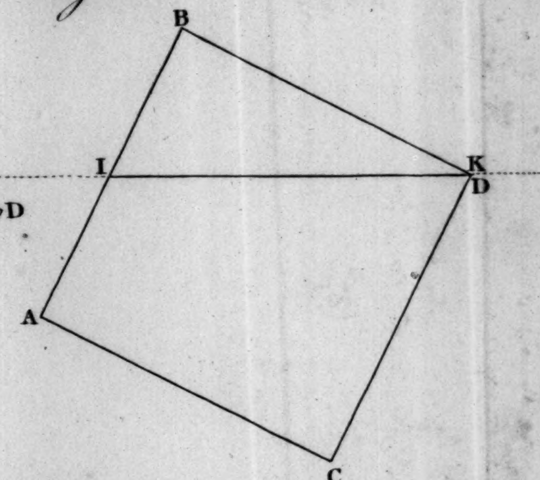

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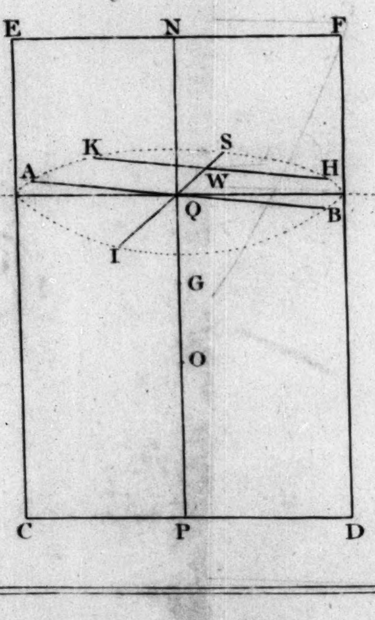

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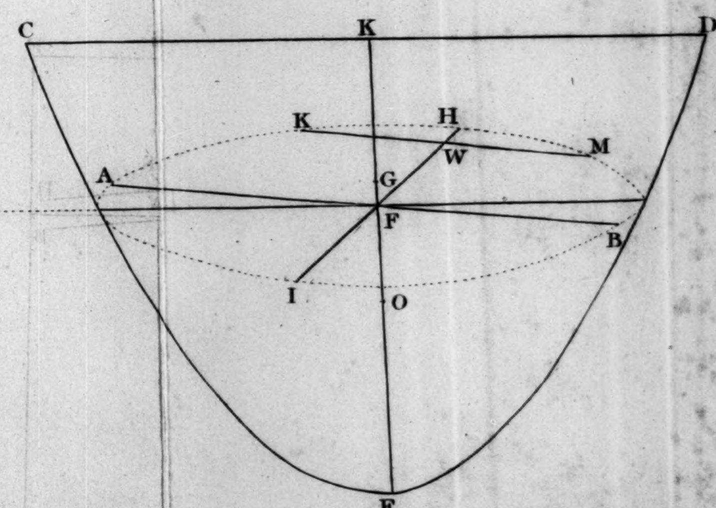

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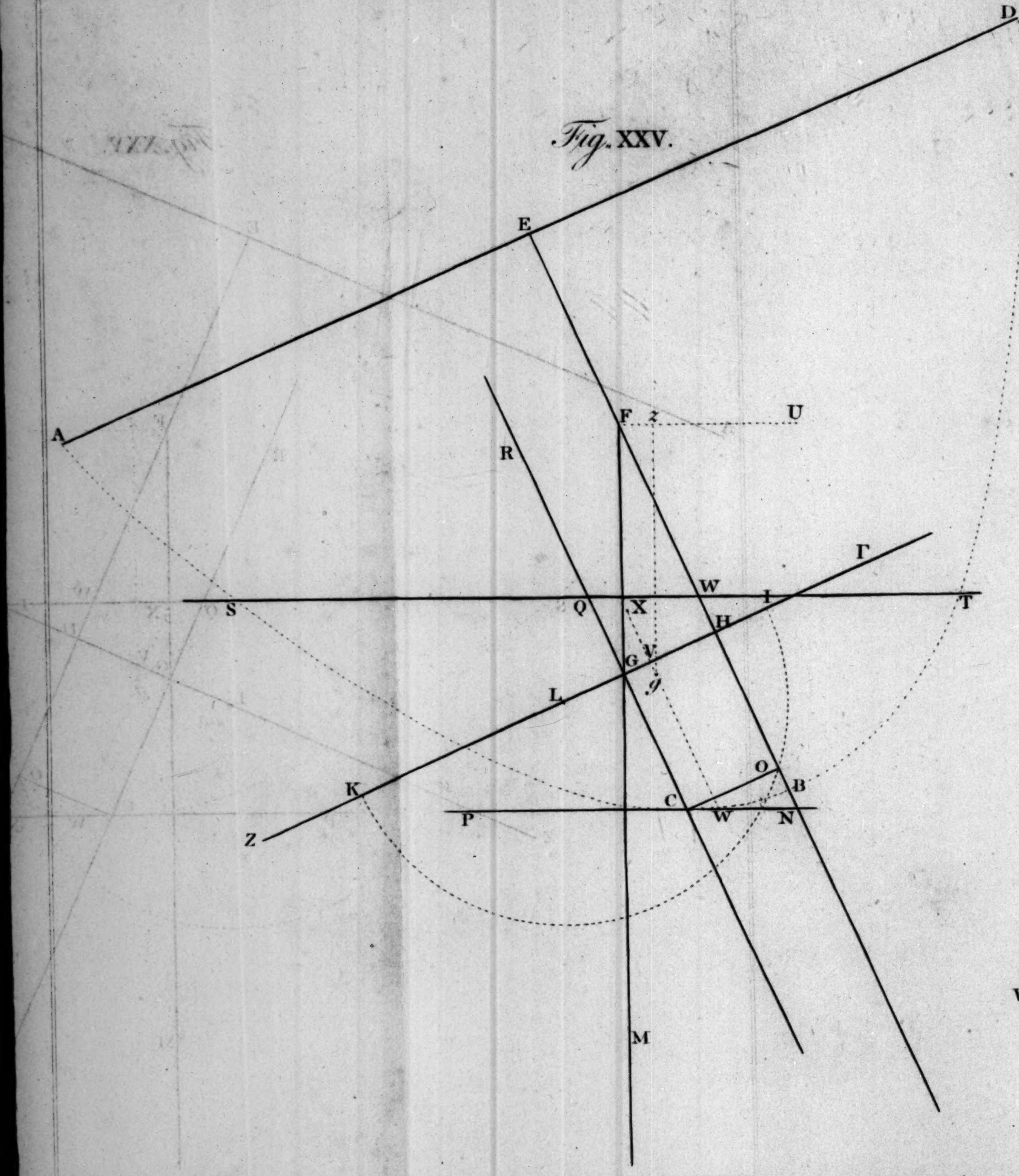

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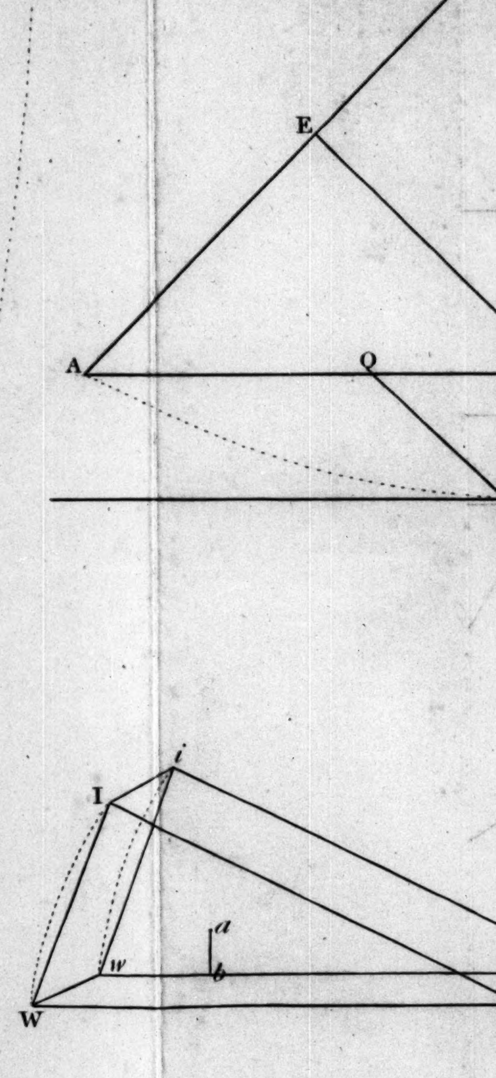

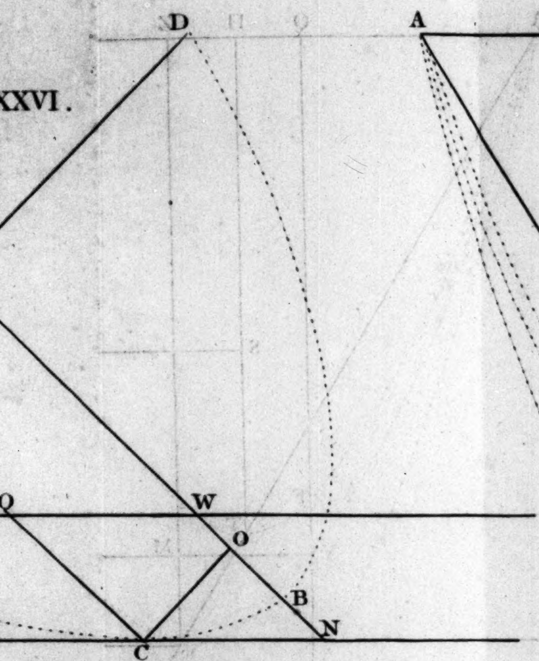

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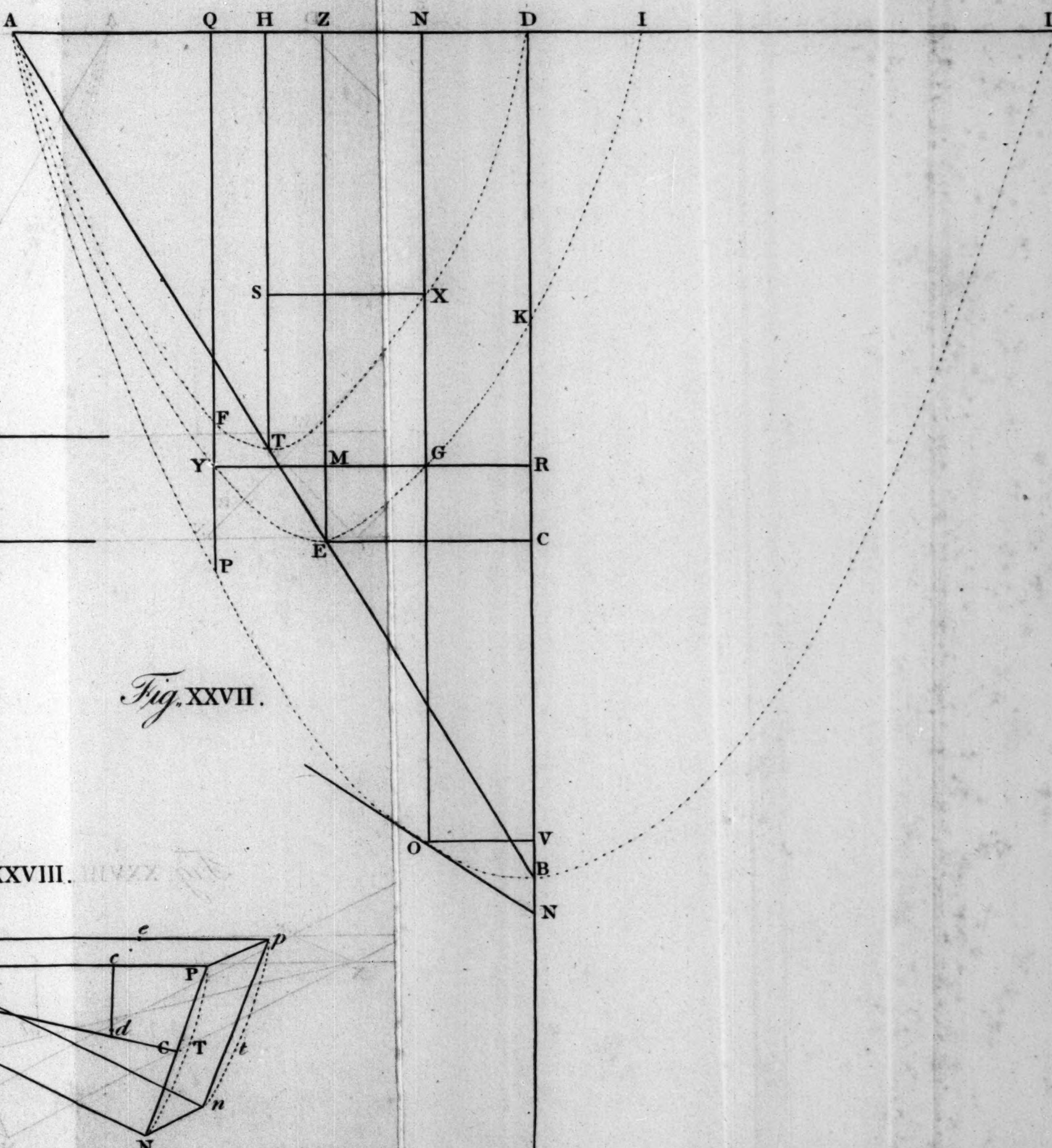

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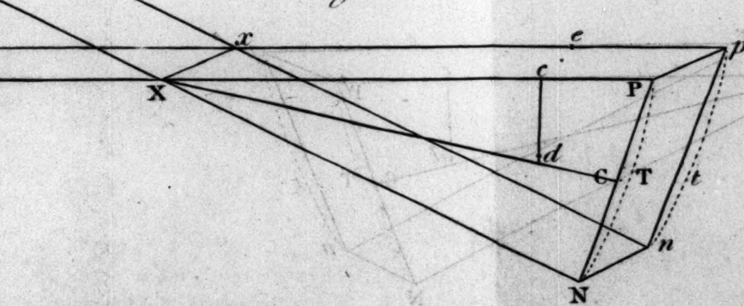

Fig. XXVIII.

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